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Edited by

C. TRUESDELL

Volume 5, Number 1

SPRINGER-VERLAG
BERLIN-GÖTTINGEN-HEIDELBERG
(Postverlagsort Berlin · 25. 4. 1960)

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Mechanicam vero duplicem Veteres constituerunt: Rationalem *quae per Demonstrationes accurate procedit, & Practicam. Ad practicam spectant Artes omnes Manuales a quibus utique* Mechanica *nomen mutuata est. Cum autem Artifices parum accurate operari soleant, fit ut* Mechanica *omnis a* Geometria *ita distinguatur, ut quicquid accuratum sit ad* Geometriam *referatur, quicquid minus accuratum ad* Mechanicam. *Attamen errores non sunt Artis sed Artificum. Qui minus accurate operatur, imperfectior est Mechanicus, & si quis accuratissime operari posset, hic foret Mechanicus omnium perfectissimus.* NEWTON

La généralité que j'embrasse, au lieu d'éblouïr nos lumieres, nous découvrira plutôt les véritables loix de la Nature dans tout leur éclat, & on y trouvera des raisons encore plus fortes, d'en admirer la beauté & la simplicité. EULER

Ceux qui aiment l'Analyse, verront avec plaisir la Méchanique en devenir une nouvelle branche ... LAGRANGE

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5856-24A

Eigenfunction Expansions Associated with the Schroedinger Operators and their Applications to Scattering Theory

TERUO IKEBE

Communicated by C. MÜLLER

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Introduction

In the present paper we shall be concerned with eigenfunction expansions associated with the Schroedinger operator $-\Delta+q(x)$, where Δ denotes the 3-dimensional Laplacian and $q(x)$ is a real-valued potential function defined on the whole 3-dimensional Euclidean space $E=E^3$, which tends to 0 at infinity. Solving the expansion problem, we go forward to apply the results obtained to clarify some properties of the spectrum of the Schroedinger operator and to show the unitary character of the S-matrix that plays an important part in describing the scattering process in quantum mechanics.

So far as ordinary differential equations are concerned, the eigenfunction expansion theory that originated from a study of WEYL [*36*] has been developed to a satisfactory extent by the efforts of STONE [*31*], TITCHMARSH [*34*], KODAIRA [*19*], YOSIDA [*38*], KREIN, LEVITAN and others [1].

As for partial differential equations, however, it seems that no complete theory, comparable with the one for ordinary differential equations, has been presented. One of the main difficulties appears to consist in the presence of infinite multiplicities of the spectrum, which is not the case with ordinary differential equations.

In 1934 CARLEMAN [*6*] studied the Schroedinger operator in E under the assumption that $q(x)$ is locally square integrable, and obtained the following result: If the unique self-adjoint extension of the operator $-\Delta+q(x)$ exists, then there exist functions $\vartheta(x, y; \mu)$, called *spectral functions*, such that for any $f(x) \in L_2(E)$ the expansion formula

$$f(x) = \int_{-\infty}^{\infty} d_\mu \int_E \vartheta(x, y; \mu) f(y)\, dy \tag{1}$$

holds, where the formal derivatives with respect to μ of $\vartheta(x, y; \mu)$ satisfy the Schroedinger equation

$$-\Delta\varphi + q(x)\varphi = \mu\varphi \tag{2}$$

as functions of x and also of y. The same problem was investigated also by TITCHMARSH, mainly in the 2-dimensional case [2]. POVZNER [*28*] extended CARLEMAN's result to every self-adjoint extension of the operator $-\Delta+q(x)$ and, moreover, using the Radon-Nikodym theorem, proved the existence of the *spectral density* $\varrho(\mu)$, by which $\vartheta(x, y; \mu)$ can be represented as

$$d_\mu \vartheta(x, y; \mu) = \psi(x, y; \mu)\, d\varrho(\mu).$$

Thus (1) reduces to

$$f(x) = \int_{-\infty}^{\infty} d\varrho(\mu) \int_E \psi(x, y; \mu) f(y)\, dy, \tag{3}$$

where $\psi(x, y; \mu)$ serve as eigenfunctions, *i.e.* they satisfy (2) as functions of x and also of y. But these eigenfunctions are not separated in the form of linear combinations of the products of the eigenfunctions of *one space-variable*, whereas this is the case with the ordinary differential equations. Such a separation of $\psi(x, y; \mu)$ was effected by MAUTNER [*21*] and GÅRDING [*9*] in the form

$$f(x) = \int_{-\infty}^{\infty} \sum_{\nu=1}^{N_\mu} \psi_\nu(x, \mu)\, d\varrho(\mu) \int_E \overline{\psi_\nu(y, \mu)} f(y)\, dy, \tag{4}$$

where $\psi_\nu(x, \mu)$ are eigenfunctions ($N_\mu \leq \infty$).

Other approaches to eigenfunction expansions connected with more general partial differential operators were made by MAUTNER [*21*], GÅRDING [*9*], BROWDER [*2, 3, 4*], GEL'FAND and KOSTYUCHENKO [*10*], BEREZANSKI [*1*], FLEKSER [*8*] and ITO [*13*]. Their results can be summed up in the form (4) in the case of the Schroedinger operator in E.

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[1] See TITCHMARSH [*34, 35*], NAIMARK [*27*] and the references cited in these books.

[2] See [*35*]. It is also abundant in references up to about 1956.

We should remark here that these authors have not specified the eigenfunctions needed for expansion. In other words, a prescription is wanting for the conditions under which we should solve (2) in order to obtain the eigenfunctions $\psi_\nu(x, \mu)$ in question. Another point to be noted is that in (4) a countable number of eigenfunctions associated with a given spectral point are used, while a continuous family of such functions is also known to exist in simple examples.

We shall consider the case where $q(x)$ tends to 0 at infinity. More precisely, we assume that $q(x)$ is Hoelder continuous except for a finite number of singularities, is square integrable and behaves like $O(|x|^{-2-h})$ $(h>0)$ as $|x| \to \infty$. Many an important problem of potential scattering is included in this case. Now we can find bounded solutions $\varphi(x, k)$ of the Schroedinger equation (2) for $\mu>0$, where k denotes a 3-dimensional *wave vector* such that $|k|^2=\mu$, as unique solutions of the integral equation

$$\varphi(x, k) = e^{i k \cdot x} - \frac{1}{4\pi} \int\limits_E \frac{e^{i|k|\,|x-y|}}{|x-y|} q(x)\, \varphi(y, k)\, dy, \tag{5}$$

$k \cdot x$ denoting the scalar product of k and x. $\varphi(x, k)$ represents the *distorted plane wave, i.e.* the plane wave plus the outgoing scattered wave. On the other hand, in case of negative μ, (2) is solvable only for particular values of μ, called *eigenvalues*, which we denote by μ_n $(n=1, 2, \ldots)$. The associated solutions $\varphi_n(x) \in L_2(E)$ are called *eigenvectors*, to be distinguished from the eigenfunctions $\varphi(x, k)$[3]. Here we agree to count μ_n as many times as its multiplicity if μ_n is degenerate. The $\varphi_n(x)$ can be regarded as forming an orthonormal system. In terms of the eigenfunctions $\varphi(x, k)$ and the eigenvectors $\varphi_n(x)$ our expansion formula[4] reads:

$$f(x) = (2\pi)^{-\frac{3}{2}} \int\limits_M \varphi(x, k)\, \hat{f}(k)\, dk + \sum_{n=1}^{\infty} \hat{f}_n\, \varphi_n(x), \tag{6}$$

where

$$\hat{f}(k) = (2\pi)^{-\frac{3}{2}} \int\limits_E \overline{\varphi(x, k)}\, f(x)\, dx, \qquad \hat{f}_n = \int\limits_E \overline{\varphi_n(x)}\, f(x)\, dx$$

and M is the 3-dimensional space formed by vectors k, which is not essentially different from E. Formula (6) is a natural generalization of the ordinary Fourier expansion, whereas (4) is not. (6) shows that the system of functions $\varphi(x, k)$ and $\varphi_n(x)$ is *complete*, but it will also be shown that these functions form an orthonormal set (in a sense to be specified later).

Now we shall turn to the problem concerning the *S-matrix* that is known as a physical quantity which governs a scattering process[5]. The existence of the S-matrix, however, is not self-evident from the mathematical point of view. In order to study the S-matrix S, it is convienent to introduce the isometric operators $W_\pm$, called *wave operators*, by $W_\pm = \lim\limits_{t \to \pm\infty} e^{itH} e^{-itH_0}$, where H and H_0 are, in our case, $-\Delta + q(x)$ and $-\Delta$ respectively. S can be defined in terms of $W_\pm$ as $S = W_+^* W_-$ and is generally believed to be unitary.

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[3] We agree to call $\varphi(x, k)$ an *eigenfunction* belonging to the *eigennumber* $|k|$. *Cf.* also SHNOL' [*30*].

[4] For more precise expression see Theorem 5 (§ 8).

[5] See *e.g.* MØLLER [*26*].

A mathematical formulation of the S-matrix theory has been proposed by Jauch [*14*] and Kuroda [*20*]. According to their results, the existence and the unitary character of S will be cleared up if we prove the existence of $W_\pm$ and that $W_\pm$ have *one and the same range.* The existence problem of the wave operators has been investigated by Cook [*7*], Jauch & Zinnes [*15*] and Hack [*11*]. Cook assumed that $q(x) \in L_2(E)$. Jauch & Zinnes treated a spherically symmetric potential: $q(x) = |x|^{-c}$ $(1 < c < \frac{3}{2})$. Hack considered the case where $q(x)$ is locally square integrable and $q(x) = O(|x|^{-c})$ $(c > 1)$ at infinity. No results, however, have been reported by them on the unitary property of S. Recently Kuroda [*20*] has solved both the problems under the assumption that $q(x) \in L_1(E) \cap L_2(E)$. We shall give a proof that S is unitary under the conditions stated above; these conditions are weaker than Kuroda's assumption in one respect. Our proof depends partly on the so-called *time-dependent theory* and partly on the eigenfunction expansion.

We shall outline here the contents of the present paper. In I (§§ 1, 2, 3, 4 and 5) we give some relations between the resolvent of H and its kernel function, whose conjugate Fourier transform is deeply connected with the eigenfunction $\varphi(x, k)$, and introduce the kernel equations and investigate their properties. II (§§ 6—9) deals with the eigenfunction expansion. § 6 is of a preparatory character, where the eigenfunctions $\varphi(x, k)$, tools for expansion, will be introduced. In § 7 we shall comment on some properties of the spectrum of the Schroedinger operator and prove that the negative part of the spectrum consists only of discrete eigenvalues. In §§ 8 and 9 we shall state the expansion theorem with a partial proof thereof and show the absolute continuity of the positive part of the spectrum. In III (§§ 10 and 11) we shall show that S-matrix is unitary and at the same time complete the proof of the expansion theorem given in § 8. IV is reserved for remarks on the 2-dimensional and higher-dimensional cases.

I. Resolvent kernel

§ 1. Assumptions. The resolvent kernel. We shall consider the Schroedinger operator $-\Delta + q(x)$ with the potential function $q(x)$ defined on $E = E^3$, where x denotes a position vector in E, its length being $|x|$. Throughout the present paper $q(x)$ is assumed to satisfy the following conditions:

(A) *$q(x)$ is a real-valued function which is locally Hoelder continuous except at a finite number of singularities. Furthermore, $q(x)$ is square integrable $\big(q(x) \in L_2(E)\big)$ and behaves like $O(|x|^{-2-h})$ $(h>0)$ at infinity,* i.e. *there exist positive numbers h, C_0 and R_0 such that*

$$|q(x)| \leqq C_0 |x|^{-2-h} \quad \textit{for} \quad |x| \geqq R_0.$$

Let us first define the operator A by $Af(x) = -\Delta f(x) + q(x) f(x)$ for $f \in C_0^\infty(E)$, where $C_0^\infty(E)$ consists of all functions which are infinitely differentiable and have compact carriers. $C_0^\infty(E)$ is contained and dense in the Hilbert space $L_2(E)$ in the sense of the L_2-norm $\| \ \|$ ($\| \ \|$ will be used exclusively for the L_2-norm). Then it is known[6] that, under the conditions (A) imposed on $q(x)$, A is lower semi-bounded and essentially self-adjoint in $L_2(E)$. Moreover, if we denote by

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[6] See Kato [*16*], Stummel [*32*] and Wienholts [*37*]. Our conditions (A) are more stringent than required for essential self-adjointness.

H the unique self-adjoint extension of A and by H_0 the corresponding operator for the case $q(x)\equiv 0$, then it is known[7] that $D(H)=D(H_0)$.

We shall study the resolvent $R_\lambda=(H-\lambda)^{-1}$ of H and its kernel $G(x, y; \lambda)$ for non-real λ; later this kernel will be extended to real λ outside the point spectrum of H. The main results are summarized in

Theorem 1[8]. *Let* $\operatorname{Im}\lambda \neq 0$[9]. i) *$R_\lambda$ is an integral operator of the Carleman type*[10] *and its kernel $G(x, y; \lambda)$, called the resolvent kernel, satisfies the integral equation*

$$G(x, y; \lambda)=\frac{e^{i\sqrt{\lambda}|x-y|}}{4\pi|x-y|}-\frac{1}{4\pi}\int_E \frac{e^{i\sqrt{\lambda}|x-z|}}{4\pi|x-z|}\, q(z)\, G(z, y; \lambda)\, dz\,^{11} \tag{1.1}$$

as a function of x a.e.[12] *in E for* a.e. *fixed $y\in E$.* ii) *If $F(x, y; \lambda)$ is a solution of* (1.1) *such that $F(\cdot, y; \lambda)\in L_2(E)$ for each fixed y, then $F(x, y; \lambda)$ is the resolvent kernel of R_λ: $F(x, y; \lambda)=G(x, y; \lambda)$ for* a.e. *x, y in $E\times E$.*

We shall call (1.1) the *kernel equation.*

Now let B be the Banach space of all continuous functions $u(x)$ defined on E, tending uniformly to 0 as $|x|\to\infty$, with the norm

$$\|u\|_B=\max_{x\in E}|u(x)|.\,^{13}$$

We also put for $\operatorname{Im}\varkappa\geqq 0$

$$\begin{aligned} A^{(i)}(x, y; \varkappa) &= -\frac{1}{4\pi}\int_E \frac{e^{i\varkappa|x-z|}}{|x-z|}\, q(z)\, A^{(i-1)}(z, y; \varkappa)\, dz, \\ A^{(0)}(x, y; \varkappa) &= \frac{e^{i\varkappa|x-y|}}{4\pi|x-y|} \qquad (i=1, 2, 3, 4). \end{aligned} \tag{1.2}$$

Then we have the following

Theorem 2. *$A^{(i)}(x, y; \varkappa)$ are continuous in x for $x\neq y$ and $A^{(4)}(\cdot, y; \varkappa)\in B$* ($\operatorname{Im}\varkappa\geqq 0$). *Further, let* $\operatorname{Im}\varkappa>0$ *and* $\operatorname{Im}\varkappa^2\neq 0$. *Then there exists a unique solution $H^{(4)}(x, y; \varkappa)$ in B of the modified kernel equation*

$$H^{(4)}(x, y; \varkappa)=A^{(4)}(x, y; \varkappa)-\frac{1}{4\pi}\int_E \frac{e^{i\varkappa|x-z|}}{|x-z|}\, q(z)\, H^{(4)}(z, y; \varkappa)\, dz \tag{1.3}$$

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[7] See Kato [*16*]. We denote by $D(T)$ the domain of the operator T.

[8] For the proof of Theorem 1, i) it suffices to assume only that $q(x)\in L_2(E)$.

[9] $\operatorname{Im}\lambda$ means the imaginary part of λ.

[10] An integral operator T as well as its kernel $T(x, y)$ is said to be of the Carleman type if $T(x, y)$ satisfies the conditions $\int_E |T(x, y)|^2 dy<\infty$ for a.e. x and $\int_E |T(x, y)|^2 dx<\infty$ for a.e. y. See also Stone [*31*]. Povzner [*28*] showed that if $q(x)$ is continuous on E, then the resolvent R_λ of every self-adjoint extension of A is of the Carleman type. This result was extended by Burnat [*5*] so as to include the case where $q(x)$ has a countable number of singularities x_n, in the neighbourhoods of which

$$q(x)=O(|x-x_n|^{-c_n}) \qquad (c_n<\tfrac{3}{2}).$$

[11] Here and in the sequel by $\sqrt{\lambda}$ is meant the branch of the square root of λ with $\operatorname{Im}\sqrt{\lambda}>0$.

[12] "a.e." means "almost every" or "almost everywhere".

[13] Our Banach space is somewhat different from the one introduced by Povzner [*28*]. He used the Banach space B_y (y being fixed) of all functions $u(x)=a|x-y|^{-1}+v(x)$, where a is a constant and $v(x)$ is continuous and tends uniformly to 0 as $|x|\to\infty$, with the norm $\|u\|_{B_y}=|a|+\max_{x\in E}|v(x)|$. *Cf.* also Burnat [*5*].

for each fixed $y \in E$. If we put

$$(1.4) \qquad H(x, y; \varkappa) = \sum_{i=0}^{3} A^{(i)}(x, y; \varkappa) + H^{(4)}(x, y; \varkappa),$$

then $H(x, y; \varkappa) = G(x, y; \varkappa^2)$ for a.e. *x, y in $E \times E$ and, moreover,*

$$(1.5) \qquad \|H(\cdot, y; \varkappa)\| \leqq C,$$

where C is a constant dependent on $\varkappa$ but not on $y \in E$.

As a consequence of Theorem 2, we can regard the resolvent kernel $G(x, y; \lambda) = H(x, y; \sqrt{\lambda})$ as a function of x defined *everywhere* in E except at $x = y$ for *each* fixed y.

§ 2. Proof of Theorem 1. The kernel equation. Let us first show that R_λ is an integral operator of the Carleman type. Denoting by $R_{0\lambda}$ the resolvent of H_0 and by V the operator of multiplication by $q(x)$, we have the following operator equations (the so-called second resolvent equations[14]) in virtue of the relation $D(H) = D(H_0)$:

$$(2.1) \qquad R_\lambda - R_{0\lambda} = -R_{0\lambda} V R_\lambda \quad \text{and} \quad R_\lambda - R_{0\lambda} = -R_\lambda V R_{0\lambda}.$$

$R_{0\lambda}$ is known to be an integral operator of the Carleman type with the kernel $(4\pi|x-y|)^{-1} e^{i\sqrt{\lambda}|x-y|}$. Let us put $\sqrt{\lambda} = a + ib$ $(b > 0)$. Then we have

$$\int_E \int_E \left| q(x) \frac{e^{i\sqrt{\lambda}|x-y|}}{4\pi|x-y|} \right|^2 dx\, dy = C \int_E |q(x)|^2 dx \int_E \frac{e^{-2b|x-y|}}{|x-y|^2} dy = C\|q\|^2;$$

here and henceforth we denote by C any constant, not necessarily the same (C in the second member of the above equality is a pure constant and C in the third depends on b). This implies that $VR_{0\lambda}$ is a completely continuous operator of the Hilbert-Schmidt type. R_λ being a bounded operator, $R_\lambda V R_{0\lambda}$ is also of the Hilbert-Schmidt type and should be an integral operator with a Hilbert-Schmidt kernel[15], say $K(x, y; \lambda)$. Thus we have

$$R_\lambda V R_{0\lambda} f(x) = \int_E K(x, y; \lambda) f(y)\, dy \quad \text{a.e.},$$

where

$$\int_E \int_E |K(x, y; \lambda)|^2 dx\, dy < \infty.$$

Since $R_{0\lambda}$ and $R_\lambda V R_{0\lambda}$, *a fortiori* by the above argument, are of the Carleman type, we see from (2.1) that R_λ is of the same type, too, and is representable as

$$(2.2) \qquad R_\lambda f(x) = \int_E G(x, y; \lambda) f(y)\, dy \quad \text{a.e.}$$

by means of the *resolvent kernel* $G(x, y; \lambda)$, where

$$\int_E |G(x, y; \lambda)|^2 dy < \infty \quad \text{for a.e. } x, \qquad \int_E |G(x, y; \lambda)|^2 dx < \infty \quad \text{for a.e. } y.$$

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[14] See Hille & Phillips [*12*], p. 126.

[15] See *e.g.* Riesz & Nagy [*29*], Chapter VI.

Let us proceed to the deduction of (1.1). For any $f(x) \in L_2(E)$ we obtain from (2.1) and (2.2) the equation

$$\begin{aligned}\int_E G(x, y; \lambda) f(y)\, dy = \frac{1}{4\pi} \int_E \frac{e^{i\sqrt{\lambda}|x-y|}}{|x-y|} f(y)\, dy - \\ - \frac{1}{4\pi} \int_E G(x, z; \lambda)\, q(z)\, dz \int_E \frac{e^{i\sqrt{\lambda}|z-y|}}{|z-y|} f(y)\, dy \quad \text{a.e.}\end{aligned} \tag{2.3}$$

By repeated use of SCHWARZ's inequality we have

$$\int_E |G(x, z; \lambda)\, q(z)|\, dz \int_E \left| \frac{e^{i\sqrt{\lambda}|z-y|}}{|z-y|} f(y) \right| dy \leqq C \|G(x, \cdot; \lambda)\|\, \|q\|\, \|f\| < \infty$$

for a.e. x. We can, therefore, interchange the order of integration in the last term of (2.3) by FUBINI's theorem. In view of the arbitrariness of $f \in L_2(E)$, it follows that

$$G(x, y; \lambda) = \frac{e^{i\sqrt{\lambda}|x-y|}}{4\pi|x-y|} - \frac{1}{4\pi} \int_E G(x, z; \lambda)\, q(z) \frac{e^{i\sqrt{\lambda}|z-y|}}{|z-y|}\, dz$$

for a.e. y with a.e. fixed x. By use of Lemma 1.1 just below we can replace $G(x, y; \lambda)$ by $G(y, x; \lambda)$ and $G(x, z; \lambda)$ by $G(z, x; \lambda)$ and, on interchanging x and y, we obtain (1.1). Thus we have proved i) of Theorem 1.

Lemma 2.1. *$G(x, y; \lambda)$ is symmetric in x and y:*

$$G(x, y; \lambda) = G(y, x; \lambda) \quad \textit{for a.e. } x, y \textit{ in } E \times E. \tag{2.4}$$

Proof. Let us introduce an indefinite "inner product" $\langle\, , \rangle$ in $L_2(E)$ by $\langle f, g\rangle = \int_E f(x)\, g(x) = (f, \bar{g})$. Then what we have to show is that R_λ is symmetric with respect to this inner product, *i.e.* $\langle R_\lambda f, g\rangle = \langle f, R_\lambda g\rangle$. For this purpose it suffices to show that $\langle Hf, g\rangle = \langle f, Hg\rangle$. But this is obvious, since we have $\langle Hf, g\rangle = \langle H_0 f, g\rangle + \langle Vf, g\rangle = (H_0 f, \bar{g}) + \langle f, Vg\rangle = (f, \overline{H_0 g}) + \langle f, Vg\rangle = \langle f, H_0 g\rangle + \langle f, Vg\rangle = \langle f, Hg\rangle$, noting that H_0 is a real operator and that V is an operator of multiplying by $q(x)$.

Let us now prove ii) of Theorem 1. To this end consider the difference $f(x) = F(x, y; \lambda) - G(x, y; \lambda)$ which is by assumption in $L_2(E)$ (for a.e. fixed y) and satisfies the equation

$$f(x) = -\frac{1}{4\pi} \int_E \frac{e^{i\sqrt{\lambda}|x-z|}}{|x-z|} q(z) f(z)\, dz \quad \text{a.e.} \tag{2.5}$$

The kernel $(4\pi|x-z|)^{-1} e^{i\sqrt{\lambda}|x-z|} q(z)$ is of the Hilbert-Schmidt type (see above) and hence defines a completely continuous operator T, which is an extension of $R_{0\lambda} V$. Therefore, we can rewrite (2.5) in the form

$$f = -Tf \quad \text{or} \quad f = (-T)^n f \qquad (n = 1, 2, \ldots), \tag{2.6}$$

where T^n are also integral operators of the Hilbert-Schmidt type. Consequently we have, noting (1.2),

$$f(x) = -\int_E A^{(3)}(x, z; \sqrt{\lambda})\, q(z)\, f(z)\, dz.$$

The function $A^{(3)}(x, z; \sqrt{\lambda})$ is bounded in x and z, as will be shown in Lemma 2.2 that follows. This fact together with SCHWARZ's inequality gives

$$|f(x)| \leqq C \int_E |q(z) f(z)| \, dz \leqq C \|q\| \, \|f\|.$$

Thus $f(x)$ is bounded and hence $Vf \in L_2(E)$, which permits us to rewrite (2.6) as

$$f = -R_{0\lambda} V f \in D(H_0). \tag{2.7}$$

Application of $H_0 - \lambda$ to both sides of (2.7) yields

$$(H_0 - \lambda) f = -Vf \quad \text{or} \quad Hf = \lambda f. \tag{2.8}$$

Since H is self-adjoint and λ is non-real, we have $f = 0$, which proves ii) of Theorem 1.

Lemma 2.2. *If* $\operatorname{Im}\varkappa \geqq 0$, *then* $A^{(3)}(x, y; \varkappa)$ *and* $A^{(4)}(x, y; \varkappa)$ *are bounded in* x, y *and* $\varkappa$.

Proof. Let us show first that $A^{(1)}(x, y; \varkappa) = O(|x-y|^{\frac{1}{2}})$. By means of SCHWARZ's inequality we have for $|x-y| = \delta > 0$

$$|A^1(x, y; \varkappa)|^2 \leqq C \int_E \frac{dz}{|x-z|^2 \, |z-y|^2} \|q\|^2 = C/\delta$$

in consideration of homogeneity of the above integral in $x-y$. Thus we have

$$|A^{(1)}(x, y; \varkappa)| \leqq C\, |x-y|^{-\frac{1}{2}}, \tag{2.9}$$

where C depends neither on x, y nor on $\varkappa$.

Next we show that $A^{(2)}(x, y; \varkappa) = O\big(|\log|x-y||\big)$. We have

$$|A^{(2)}(x, y; \varkappa)| \leqq C \left(\int_{K(R_0)} \frac{|q(z)|}{|x-z|} |A^{(1)}(z, y; \varkappa)| \, dz + \int_{K(R_0)'} \right) = J + J'.$$

Here and in the sequal $K(x, R)$ denotes the sphere of radius R with its centre at $x\,(K(R) = K(0, R))$ and $K(x, R)' = E - K(x, R)$ the exterior of K_R. (For R_0 see (A) of § 1). We also denote by $K(R) \frown K(R_1)$ the intersection of $K(R)$ and $K(R_1)$. We shall now estimate J' as follows: By (2.9) and (A) we get

$$J' \leqq C \left(\int_{K(R_0)' \frown K(y,1)} \frac{dz}{|z|^{2+h} |x-z| \, |y-z|^{\frac{1}{2}}} + \int_{K(R_0)' - K(y,1)} \right) = J_1' + J_2'.$$

An estimation of J_1' is

$$J_1' \leqq C \int_{K(y,1)} \frac{dz}{R_0^{2+h} |x-z| \, |y-z|^{\frac{1}{2}}} \leqq C \left[\int_{K(y,1)} \frac{dz}{|x-z|^2} \int_{K(y,1)} \frac{dz}{|y-z|} \right]^{\frac{1}{2}} \leqq C.$$

For J_2' we have, introducing spherical coordinates,

$$J_2' \leqq C \int\limits_{K(R_0)'} \frac{dz}{|z|^{2+h}\,|x-z|} = C \int\limits_{R_0}^{\infty}\int\limits_{0}^{\pi} \frac{r^2 \sin\vartheta \, dr \, d\vartheta}{r^{2+h}(|x|^2+r^2-2|x|\,r\cos\vartheta)^{\frac{1}{2}}}$$

$$= C\,|x|^{-1} \int\limits_{R_0}^{\infty} r^{-1-h}\big(|x| + r - \big||x| - r\big|\big)\, dr.$$

If $|x| \leqq R_0$ and $x \neq 0$, we obtain

$$J_2' \leqq C \int\limits_{R_0}^{\infty} r^{-1-h}\, dr \leqq C;$$

the case $x=0$ can be dealt with separately, and a similar result follows. If $|x| > R_0$

$$J_2' \leqq C\,|x|^{-1} \int\limits_{R_0}^{|x|} r^{-h}\, dr + C\,|x|^{-1} \int\limits_{|x|}^{\infty} |x|\, r^{-1-h}\, dr \leqq C.$$

Thus J_2' is always bounded; hence so is J'.

Now we have to estimate J:

$$J^2 \leqq C \int\limits_{K(R_0)} \frac{dz}{|x-z|^2\,|y-z|} \int\limits_{K(R_0)} |q(z)|^2\, dz$$

$$= C\Bigg(\int\limits_{K(R_0)\cap K(x,\,\delta/2)} \frac{dz}{|x-z|^2\,|y-z|} + \int\limits_{K(R_0)-K(x,\,\delta/2)}\Bigg) = J_1 + J_2 \qquad (\delta = |x-y|),$$

$$J_1 \leqq C\,\delta^{-1} \int\limits_{K(x,\,\delta/2)} \frac{dz}{|x-z|^2} \leqq C.$$

If $\delta \geqq \frac{1}{2}$, we have for J_2

$$J_2 \leqq C \int\limits_{K(R_0)} \frac{dz}{|x-z|^2\,|y-z|} \leqq C \int\limits_{K(x,\,\frac{1}{4})} \frac{dz}{|x-z|^2} + C \int\limits_{K(y,\,\frac{1}{4})} \frac{dz}{|y-z|} + C \int\limits_{K(R_0)} dz \leqq C.$$

If $\delta < \frac{1}{2}$, we first assume that $|x| \geqq R_0 + 1$. Then

$$J_2 \leqq C \int\limits_{K(R_0)} \frac{1}{|y-z|}\, dz \leqq C.$$

For $|x| < R_0 + 1$, on translating the origin, we have

$$J_2 \leqq C \int\limits_{K(2R_0+1)-K(\delta/2)} \frac{dz}{|z|^2\,|x-y-z|} = C\,\delta^{-1} \int\limits_{\delta/2}^{R_0+1} r^{-1}(r + \delta - |r-\delta|)\, dr \leqq C\,|\log\delta|.$$

Thus we get

$$(2.10) \qquad \begin{aligned} |A^2(x,y;\varkappa)| &\leqq C \quad \text{if} \quad |x-y| \geqq \tfrac{1}{2} \\ &\leqq C\,\big|\log|x-y|\big| \quad \text{if} \quad |x-y| < \tfrac{1}{2} \end{aligned}$$

where C is independent of x, y and $\varkappa$.

Now we shall show that $A^{(3)}(x, y; \varkappa)$ is bounded. We have from (2.10)

$$|A^{(3)}(x, y; \varkappa)| \leqq C \int\limits_{K(y, \frac{1}{2})} \frac{|q(z)|}{|x-z|} |\log|z-y|| \, dz + C \int\limits_{K(y, \frac{1}{2})'} \frac{|q(z)|}{|x-z|} dz = L + L'.$$

L and L' can be estimated as follows:

$$L \leqq C \Big[\int\limits_{K(y, \frac{1}{2})} \frac{|\log|z-y||^2}{|x-z|^2} dz \int\limits_{K(y, \frac{1}{2})} |q(z)|^2 dz \Big]^{\frac{1}{2}} \leqq C,$$

$$L' \leqq C \int\limits_{E} \frac{|q(z)|}{|x-z|} dz = C \Big(\int\limits_{K(R_0)} \frac{|q(z)|}{|x-z|} dz + \int\limits_{K(R_0)'} \Big) = L_1' + L_2',$$

$$L_1' \leqq C \Big[\int\limits_{K(R_0)} \frac{dz}{|x-z|^2} \int\limits_{K(R_0)} |q(z)|^2 dz \Big]^{\frac{1}{2}} \leqq C,$$

$$L_2' \leqq C \int\limits_{K(R_0)'} \frac{dz}{|z|^{2+h} |x-z|} \leqq C. \qquad \text{(See } J_2' \text{ above.)}$$

These estimates show that

$$|A^{(3)}(x, y; \varkappa)| \leqq C, \tag{2.11}$$

where C is independent of x, y and $\varkappa$.

From (2.11) and the estimation performed for L' we get

$$|A^{(4)}(x, y; \varkappa)| \leqq C, \tag{2.12}$$

C being independent of x, y and $\varkappa$. (2.11) and (2.12) prove the assertion of Lemma 2.2.

The following lemma will be needed for the proof of Theorem 2 that we shall give in § 5.

Lemma 2.3. *If* $\operatorname{Im} \varkappa > 0$, *then* $\|A^{(i)}(\cdot, y; \varkappa)\| \leqq C$ $(i = 0, 1, 2, 3, 4)$, *where* C *depends on* $\varkappa$ *but not on* y.

Proof. It is easily seen that $\|A^{(0)}(\cdot, y; \varkappa)\| \leqq C$, where C does not depend on y. We assume that this is the case for $A^{(i)}(\cdot, y; \varkappa)$ and prove that the assertion is true for $A^{(i+1)}(\cdot, y; \varkappa)$.

Using the operator T defined before (see (2.8) and (2.9)), we have $A^{(i+1)}(\cdot, y; \varkappa) = T A^{(i)}(\cdot, y; \varkappa)$, whence follows

$$\|A^{(i+1)}(\cdot, y; \varkappa)\| \leqq C \|A^{(i)}(\cdot, y; \varkappa)\| \leqq C \tag{2.13}$$

in view of the boundedness of T.

§ 3. Asymptotic behavior of the function $-\frac{1}{4\pi} \int\limits_E \frac{e^{i\varkappa|x-y|}}{|x-y|} v(y) \, dy$.

In order to solve the kernel and modified kernel equations, we shall study some asymptotic properties of functions of the form

$$q(x) = -\frac{1}{4\pi} \int\limits_E \frac{e^{i\varkappa|x-y|}}{|x-y|} v(y) \, dy \qquad (\operatorname{Im} \varkappa \geqq 0).$$

Lemma 3.1. *Let* $\operatorname{Im}\varkappa \geqq 0$, *let* $v(x)$ *be locally integrable and* $v(x) = O(|x|^{-2-\varepsilon})$ $(|x| \to \infty)$ $(\varepsilon > 0)$. *Then as* $|x| \to \infty$

$$\varphi(x) = O(|x|^{-1}) + O(|x|^{-\varepsilon}). \tag{3.1}$$

Proof. Let R be such that $|v(x)| \leqq C|x|^{-2-\varepsilon}$ for $|x| \geqq R$. If $|x| > R$ we have

$$\varphi(x) = -\frac{1}{4\pi}\int\limits_{K(R)} \frac{e^{i\varkappa|x-y|}}{|x-y|} v(y)\,dy - \int\limits_{K(R)'} = I + I',$$

$$|I'| \leqq C \int\limits_{K(R)'} \frac{|v(y)|}{|x-y|}\,dy \leqq C \int\limits_R^\infty \int\limits_0^\pi \frac{r^2 \sin\vartheta\,dr\,d\vartheta}{r^{2+\varepsilon}(|x|^2 + r^2 - 2|x|\,r\cos\vartheta)^{\frac{1}{2}}}$$

$$= C|x|^{-1} \int\limits_R^\infty r^{-1-\varepsilon}\big(|x| + r - \big||x| - r\big|\big)\,dr = C|x|^{-1} + C|x|^{-\varepsilon}.$$

In order to give an estimate for I, it suffices to note that $v(x)$ is integrable over $K(R)$ and that $|x-y| \geqq |x| - R$. Then we have $I = O(|x|^{-1})$. By putting I and I' together, (3.1) follows.

Lemma 3.2[16]. *Let* $v(x) = O(|x|^{-3-\varepsilon})$ $(|x| \to \infty)$ $(\varepsilon > 0)$, $v(x) \in L_2(E)$ *and let* $\varkappa = a$ *be real. Then we have*

$$\varphi(x) = -\frac{e^{ia|x|}}{4\pi|x|}\int\limits_E e^{-ia\omega_x \cdot y} v(y)\,dy + O(|x|^{-1-\varepsilon/2}) + O(|x|^{-2}), \tag{3.2}$$

where ω_x *denotes the unit vector with the direction of* x *and* $\omega_x \cdot y$ *denotes the scalar product of* ω_x *and* y.

Proof. Let R_1 be fixed so that $|v(x)| \leqq C|x|^{-3-\varepsilon}$ for $|x| \geqq R_1$ and let $|x|$ be so large that $|x|^{\frac{1}{2}} = R > R_1$. Then we have

$$\varphi(x) = -\frac{1}{4\pi}\Bigg(\int\limits_{K(R)} \frac{e^{ia|x-y|}}{|x-y|} v(y)\,dy + \int\limits_{K(R)'}\Bigg) = I + I'. \tag{3.3}$$

An argument similar to the one used in proving Lemma 3.1 gives

$$|I'| \leqq C|x|^{-1}R^{-\varepsilon} + C|x|^{-1-\varepsilon} = C|x|^{-1-\varepsilon/2} + C|x|^{-1-\varepsilon}.$$

We proceed to estimate I. Considering $|y| < R$ we have

$$e^{ia|x-y|} = e^{ia|x|\left[1 - \left(\omega_x \cdot \frac{y}{|x|}\right) + \eta_1\right]} \qquad (\eta_1 = O(|y|/|x|)^2),$$

$$|x-y|^{-1} = |x|^{-1}\left[1 + \left(\omega_x \cdot \frac{y}{|x|}\right) + \eta_2\right] \qquad (\eta_2 = O(|y|/|x|)^2),$$

and hence

$$\frac{e^{ia|x-y|}}{|x-y|} = \frac{1}{|x|} e^{ia|x| - ia\omega_x \cdot y + ia|x|\eta_1} + \frac{\eta_2}{|x|} e^{ia|x-y|} + \frac{1}{|x|^2}\,\omega_x \cdot y\, e^{ia|x-y|}.$$

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[16] *Cf.* Povzner [*28*]. He obtained the result in which the last two terms in (3.2) are replaced by $O\Big(|x|^{-1-\frac{\varepsilon}{2+\varepsilon}}\Big)$. Our result enables us to proceed along his line without assuming that the potential has the asymptotic form $q(x) = O(|x|^{-3.5-\varepsilon})$, which was assumed in [*28*] together with the continuous differentiability of $q(x)$.

Thus we have

$$\begin{aligned} I = & -\frac{e^{ia|x|}}{4\pi|x|}\int_E e^{-ia\omega_x\cdot y}\,v(y)\,dy + \frac{e^{ia|x|}}{4\pi|x|}\int_{K(R)'} e^{-ia\omega_x\cdot y}\,v(y)\,dy + \\ & + \frac{e^{ia|x|}}{4\pi|x|}\int_{K(R)} e^{-ia\omega_x\cdot y}(1-e^{ia|x|\eta_1})\,v(y)\,dy - \\ & - \frac{1}{4\pi|x|}\int_{K(R)} \eta_2\, e^{ia|x-y|}\,v(y)\,dy - \frac{1}{4\pi|x|^2}\int_{K(R)} \omega_x\cdot y\, e^{ia|x-y|}\,v(y)\,dy \\ = & -\frac{e^{ia|x|}}{4\pi|x|}\int_E e^{-ia\omega_x\cdot y}\,v(y)\,dy + I_1 + I_2 + I_3 + I_4. \end{aligned}$$

I_i $(i=1, 2, 3, 4)$ are estimated as follows:

$$\begin{aligned} |I_1| & \leqq C|x|^{-1}\int_{K(R)'}|v(y)|\,dy \leqq C|x|^{-1}R^{-\varepsilon} = C|x|^{-1-\varepsilon/2}, \\ |I_2| & \leqq C|x|^{-1}\int_{K(R)}\frac{|y|^2}{|x|^2}\,|x|\,|v(y)|\,dy = C|x|^{-2}\Bigl(\int_{K(R_1)}|y|^2\,|v(y)|\,dy + \int_{K(R)-K(R_1)}\Bigr) \\ & \leqq C|x|^{-2} + C|x|^{-2}R^{2-\varepsilon} = C|x|^{-2} + C|x|^{-1-\varepsilon/2}, \\ |I_3| & \leqq C|x|^{-1}\int_{K(R)}\frac{|y|^2}{|x|^2}\,|v(y)|\,dy = C|x|^{-3}\Bigl(\int_{K(R_1)}|y|^2\,|v(y)|\,dy + \int_{K(R)-K(R_1)}\Bigr) \\ & \leqq C|x|^{-3} + C|x|^{-3}R^{2-\varepsilon} = C|x|^{-3} + C|x|^{-2-\varepsilon/2}, \\ |I_4| & \leqq C|x|^{-2}\int_{K(R)}|y|\,|v(y)|\,dy = C|x|^{-2}\Bigl(\int_{K(R_1)} + \int_{K(R)-K(R_1)}\Bigr) \\ & \leqq C|x|^{-2} + C|x|^{-2}R^{1-\varepsilon} = C|x|^{-2} + C|x|^{-(3+\varepsilon)/2}. \end{aligned}$$

In these estimates the constants C may or may not depend on a, ε and $v(\cdot)$. (3.2) now follows from (3.3) and the estimates for I and I'.

Lemma 3.3. *Let $v(x)$ and $\varphi(x)$ be as in Lemma 3.2. Then $\varphi(x)$ satisfies the radiation condition*[17], i.e.

$$\lim_{|x|\to\infty}|x|\left|\frac{\partial\varphi}{\partial|x|} - ia\varphi\right| = 0. \tag{3.4}$$

For the proof see POVZNER [*28*], Chapter II, Lemma 2.

Lemma 3.4. *Let $\varphi(x)\in B$ be a solution of the integral equation*

$$\varphi(x) = -\frac{1}{4\pi}\int_E \frac{e^{ia|x-y|}}{|x-y|}\,q(y)\,\varphi(y)\,dy, \tag{3.5}$$

where a is real. Then

$$\int_E e^{-ia\omega\cdot x}\,q(x)\,\varphi(x)\,dx = 0, \tag{3.6}$$

where ω is an arbitrary unit vector.

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[17] There are definitions of the radiation condition other than the one adopted here. See *e.g.* MIRANKER [*22*] and MUELLER [*24*]. Here we follow POVZNER [*28*].

For the proof see POVZNER [28], Chapter II, Lemma 5. But it seems necessary to add some remarks. The proof relies on Lemmas 3.2 and 3.3 and on GREEN'S formula. We maintain that if $\varphi(x) \in B$ satisfies (3.5), then $\varphi(x) = O(|x|^{-1})$ $(|x| \to \infty)$, so that Lemmas 3.2 and 3.3 may be applied. In fact $\varphi(x) = O(|x|^{-1}) + O(|x|^{-h})$ by Lemma 3.1 because $q(x) = O(|x|^{-2-h})$. Hence repeated application of Lemma 3.1 to (3.5) furnishes us with the result $\varphi(x) = O(|x|^{-1})$.

Another remark applies to the use of GREEN'S formula. An explicit use of the Hoelder continuity of $q(x)$ and $\varphi(x)$ is made to obtain from (3.5) the differential equation

$$-\Delta\varphi(x) + q(x)\,\varphi(x) = a^2\varphi(x), \tag{3.7}$$

which holds for x different from the singularities of $q(x)$ (Hoelder continuity of $\varphi(x)$ follows from (3.5) and the proof of Lemma 4.1 given below). We apply GREEN'S formula to φ and $\overline{\varphi}$, which also satisfies (3.7), and get

$$0 = \int\limits_{K(R)} (\Delta\varphi\cdot\overline{\varphi} - \varphi\cdot\Delta\overline{\varphi})\,dx = \int\limits_{S(R)} \left(\overline{\varphi}\frac{\partial\varphi}{\partial n} - \varphi\frac{\partial\overline{\varphi}}{\partial n}\right) ds \tag{3.8}$$

where $S(R)$ is the surface of $K(R)$ and n is the outer normal. (3.8) is valid if $K(R)$ contains no singularity of $q(x)$, but even if singularities of $q(x)$ exist in $K(R)$, it is seen to be true. For we first exclude from $K(R)$ the singularities which are finite in number, and then by a limiting process we obtain (3.8) valid for $K(R)$, when we note that $\partial\varphi/\partial n$ as well as $\partial\overline{\varphi}/\partial n$ is $O(|x-x_n|^{-\frac{1}{2}})$ (x_n: singularities of $q(x)$), as is seen from differentiation of (3.5) under the integral sign, so that the surface integrals taken around the singularities are arbitrarily small.

The above remarks are enough to get to (3.6), following the proof by POVZNER [28].

§ 4. Solution of the kernel equation. Let us define an operator $T_\varkappa$ for functions in B by

$$T_\varkappa f(x) = -\frac{1}{4\pi}\int\limits_E \frac{e^{i\varkappa|x-y|}}{|x-y|}\,q(y)\,f(y)\,dy \qquad (\operatorname{Im}\varkappa \geqq 0,\ f \in B). \tag{4.1}$$

Lemma 4.1. *$T_\varkappa$ is a bounded linear operator on B to B* $(\operatorname{Im}\varkappa \geqq 0)$.

Proof. Let $f \in B$. Then $q(x)f(x) = O(|x|^{-2-h})$ $(|x| \to \infty)$, so Lemma 3.1 shows that $T_\varkappa f(x) \to 0$ uniformly as $|x| \to \infty$.

$T_\varkappa f(x)$ is bounded in x. In fact we obtain from (4.1)

$$|T_\varkappa f(x)| \leqq C\,\|f\|_B \int\limits_E \frac{|q(y)|}{|x-y|}\,dy.$$

The boundedness in x of the right member follows from the estimation carried out for L' in the proof of Lemma 2.2.

Next we have to show the continuity in x of $T_\varkappa f(x)$, and for this purpose we consider the difference

$$\begin{aligned} T_\varkappa f(x) - T_\varkappa f(x') = &-\frac{1}{4\pi}\int\limits_E \frac{e^{i\varkappa|x-y|} - e^{i\varkappa|x'-y|}}{|x-y|}\,q(y)\,f(y)\,dy - \\ &-\frac{1}{4\pi}\int\limits_E \left[\frac{1}{|x-y|} - \frac{1}{|x'-y|}\right] e^{i\varkappa|x'-y|}\,q(y)\,f(y)\,dy = J_1 + J_2. \end{aligned} \tag{4.2}$$

Considering the inequalities $|e^{i\varkappa|x-y|}-e^{i\varkappa|x'-y|}|\leqq|\varkappa|\,|x-x'|$ and

$$\left|\frac{1}{|x-y|}-\frac{1}{|x'-y|}\right|\leqq\frac{|x-x'|}{|x-y|\,|x'-y|},$$

we can give estimates for J_1 and J_2:

$$|J_1|\leqq C|x-x'|\int_E\frac{|q(y)\,f(y)|}{|x-y|}\,dy\leqq C\|f\|_B\,x-x'|,$$

$$|J_2|\leqq C|x-x'|\,\|f\|_B\int_E\frac{|q(y)|}{|x-y|\,|x'-y|}\,dy\leqq C\|f\|_B|x-x'|^{\frac{1}{2}},$$

as is seen from the estimations worked out for L' and $A^{(1)}(x, y; \varkappa)$ in the proof of Lemma 2.2. Hence it follows that

$$T_\varkappa f(x)-T_\varkappa f(x')=O(|x-x'|^{\frac{1}{2}})\quad\text{for}\quad|x-x'|\to 0. \tag{4.3}$$

(4.3) shows that $T_\varkappa f(x)$ is not merely continuous, but also Hoelder continuous in x. Consequently $T_\varkappa f$ turns out to be an element of B. This completes the proof of the lemma.

Remark. The proof given above uses only the boundedness of $f(x)$. Thus $T_\varkappa f$ belongs to B and is even Hoelder continuous if f is only bounded.

Lemma 4.2. *$T_\varkappa$ is completely continuous*[18].

Proof. Let f_n $(n=1, 2, \ldots)$ be any sequence in B such that $\|f_n\|_B\leqq 1$. We have to show that we can take out of the sequence $u_n=T_\varkappa f_n$ a subsequence convergent in the norm of B.

Re-examining the proof of Lemma 4.1, we find out firstly that $\|u_n\|_B$ are uniformly bounded with respect to n, because $\|f_n\|_B\leqq 1$, and secondly that on the same account the continuity of $u_n(x)$ is uniform with respect to n. Moreover, we see that $u_n(x)\to 0$ uniformly in n as $|x|\to\infty$. Thus $u_n(x)$ are equi-bounded and uniformly equi-continuous with respect to n in any compact domain of E. On applying the Ascoli-Arzelà theorem to $u_n(x)$, we can choose a subsequence $u_{n'}(x)$ converging uniformly to a continuous function $u(x)$ in any compact domain. Since $u_{n'}(x)\to 0$ uniformly in n' as $|x|\to\infty$, $u(x)$ also tends to 0 at infinity, and hence we have $\|u_{n'}-u\|_B\to 0$, which was to be proved.

We are now in a position to make use of the Riesz-Schauder theory of completely continuous operators in a Banach space[19]. If T is a completely continuous operator in B, the equation $f=g+Tf$ is solvable for any given $g\in B$, if and only if $f=Tf$ implies that $f=0$. Thus we have the following

Lemma 4.3. *Let $g\in B$. Then the integral equation*

$$f(x)=g(x)-\frac{1}{4\pi}\int_E\frac{e^{i\varkappa|x-y|}}{|x-y|}\,q(y)\,f(y)\,dy\qquad(\mathrm{Im}\,\varkappa\geqq 0) \tag{4.4}$$

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[18] *Cf.* also POVZNER [*28*], Chapter II, Lemma 3.

[19] See *e.g.* RIESZ & NAGY [*29*], Chapter VI or YOSIDA [*39*], § 41.

has a unique solution in B if and only if the homogeneous equation

$$f(x) = -\frac{1}{4\pi}\int_E \frac{e^{i\varkappa|x-y|}}{|x-y|^2}\, q(y)\, f(y)\, dy \tag{4.5}$$

has the unique solution $f(x) \equiv 0$.

Lemma 4.4. *If $\varkappa^2$ is real and positive, then* (4.5) *has the unique solution $f(x) = 0$.*

Proof. If $f(x)$ satisfies (4.5), (3.7) follows from the assumed Hoelder continutiy of $q(x)$ and that of $f(x)$, remarked in the proof of Lemma 4.1 (see also the remarks to Lemma 3.4). On the other hand, Lemmas 3.2 and 3.4 give the asymptotic order

$$f(x) = o(|x|^{-1}) \qquad (|x| \to \infty). \tag{4.7}$$

According to Kato [*18*], however, any solution of (3.7) subject to the condition (4.7) vanishes indentically at least outside a sufficiently large sphere containing all the singularities of $q(x)$. Hence the unique continuation theorem for solution of an elliptic differential equation[20] is applied to obtain the result that $f(x) = 0$ everywhere.

Lemma 4.5. *Let* $\operatorname{Im}\varkappa > 0$. *Then* (4.5) *has non-trivial solutions in B if and only if $\varkappa^2$ is an eigenvalue of H. In particular, if* $\operatorname{Im}\varkappa^2 \neq 0$ *in addition,* (4.5) *has only the trivial solution $f(x) \equiv 0$.*

Proof. Let $f(x) \in B$ satisfy (4.5). Since $q(x) \in L_2(E)$, $q(x) f(x) \in L_2(E)$. Then from (4.5) we have $f = -R_{0\varkappa^2} V f$, which is nothing but (2.8) with $\lambda = \varkappa^2$, and hence we get $f = 0$ if $\varkappa^2$ is non-real, in exactly the same way as in § 2. This proves the second assertion.

Let $f(x)$ be a non-trivial solution of (4.5) ($\operatorname{Im}\varkappa > 0$). Then, as we have seen before (the proof of Theorem 1, ii) in § 2), it follows that f, $Vf \in L_2(E)$ and $Hf = \varkappa^2 f$. This proves the necessity of the first assertion.

Conversely let $\varkappa^2$ be an eigenvalue of H with an eigenvector $f \in D(H) = D(H_0)$. Then $Vf \in L_2(E)$ and (4.5) readily follows. Accordingly, by use of Schwarz's inequality, we have

$$|f(x)| \leqq C\left(\int_E \frac{e^{-2b|x-y|}}{|x-y|^2}\, dy \int_E |q(y) f(y)|^2\, dy\right)^{\frac{1}{2}} \leqq C \qquad (b = \operatorname{Im}\varkappa),$$

from which and (4.5) we see that $f \in B$, as in the proof of Lemma 4.1. This proves the sufficiency.

Lemma 4.6. $T_\varkappa$ ($\operatorname{Im}\varkappa \geqq 0$) *depends continuously on $\varkappa$*; i.e. *given any $\varepsilon > 0$, there exists a $\delta = \delta(\varepsilon)$ such that $\|T_{\varkappa_1} - T_{\varkappa_2}\|_B < \varepsilon$*[21] *if $|\varkappa_1 - \varkappa_2| < \delta$.*

For the proof see Povzner [*28*], Chapter II, Lemma 8. We have adopted a norm different from the one used by Povzner (see footnote 13), but this makes the proof of the present lemma rather simpler, for we have avoided inclusion of singular functions in B.

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[20] See *e.g.* Mueller [*25*].

[21] The norm of an operator T in B is defined by $\|T\|_B = \sup_{\|f\|_B = 1} \|Tf\|_B$.

All the bounded operators on B form a Banach algebra $\boldsymbol{B}$, in which $\|T_n - T\|_B \to 0$ $(n \to \infty)$ and the existence of T_n^{-1} and $T^{-1} \in \boldsymbol{B}$ imply $\|T_n^{-1} - T^{-1}\|_B \to 0$[22]. Thus from Lemma 4.6 follows

Lemma 4.7. *If $\varkappa^2$ is not a non-positive eigenvalue of H, $(I - T_\varkappa)^{-1} \in B$ exists and depends continuously on $\varkappa$ in the sense of the norm $\| \, \|_B$.*

Lemma 4.8. *Let D be a compact domain of the upper $\varkappa$-plane* $(\operatorname{Im}\varkappa \geqq 0)$ *such that D does not contain the square roots of the non-positive eigenvalues. Let $f_\varkappa \in B$ be a unique solution of*

$$f_\varkappa = g_\varkappa + T_\varkappa f_\varkappa, \tag{4.9}$$

where $g_\varkappa \in B$ is strongly continuous in $\varkappa \in D$. Then $f \in B$ is strongly continuous in $\varkappa \in D$.

Proof. Since $f_\varkappa = (I - T_\varkappa)^{-1} g_\varkappa$ and $(I - T_\varkappa)^{-1}$ is uniformly continuous in $\varkappa \in D$ (Lemma 4.7) and also $g_\varkappa$ is by assumption strongly continuous in $\varkappa$, the assertion follows immediately.

§ 5. Proof of Theorem 2. We shall show first that $A^{(i)}(x, y; \varkappa)$ are continuous in x unless $x = y$. By definition we have for $i = 0, 1, 2, 3$

$$\begin{aligned} A^{(i+1)}(x, y; \varkappa) &- A^{(i+1)}(x', y; \varkappa) \\ &= -\frac{1}{4\pi} \int_E \frac{e^{i\varkappa|x-z|} - e^{i\varkappa|x'-z|}}{|x-z|} q(z) A^{(i)}(z, y; \varkappa)\, dz - \\ &\quad - \frac{1}{4\pi} \int_E \left[\frac{1}{|x-z|} - \frac{1}{|x'-z|}\right] e^{i\varkappa|x'-z|} q(z) A^{(i)}(z, y; \varkappa)\, dz \\ &= J_1 + J_2. \end{aligned}$$

Proceeding as in the proof of Lemma 4.1, we arrive at the following inequalities for J_1 and J_2:

$$|J_1| \leqq C|x - x'| \int_E \frac{|q(z) A^{(i)}(z, y; \varkappa)|}{|x-z|} dz,$$

$$|J_2| \leqq C|x - x'| \int_E \frac{|q(z) A^{(i)}(z, y; \varkappa)|}{|x-z|\,|x'-z|} dz.$$

Since $A^{(i)}(z, y; \varkappa)$ are bounded in z outside some sphere with its centre at y, as is seen in the proof of Lemma 2.2, we get

$$|J_1| \leqq C|x - x'| \int_{K(x,\delta)} \frac{|q(z)|}{|x-z|} dz + C|x - x'| \int_{K(x,\delta)'} |q(z) A^{(i)}(z, y; \varkappa)|\, dz,$$

$$\begin{aligned} |J_2| \leqq C|x - x'| &\int_{K(x,\delta)} \frac{|q(z)|}{|x-z|\,|x'-z|} dz + C|x - x'| \int_{K(x',\eta)} \frac{|q(z)|}{|x-z|\,|x'-z|} dz + \\ &+ C|x - x'| \int_{K(x,\delta)' \cap K(x',\eta)'} |q(z) A^{(i)}(z, y; \varkappa)|\, dz, \end{aligned}$$

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[22] See any book on Banach algebras, *e.g.* HILLE & PHILLIPS [*12*] and YOSIDA [*39*].

where we have chosen δ and η so that $K(x, \delta)$ and $K(x', \eta)$ do not contain the point y. Then from the estimations in the proofs of Lemmas 2.3 and 4.1 we obtain by SCHWARZ's inequality

$$|J_1| \leqq C|x - x'| + C|x - x'| \, \|q\| \, \|A^{(i)}(\cdot, y; \varkappa)\|,$$

$$|J_2| \leqq C|x - x'|^{\frac{1}{2}} + C|x - x'|^{\frac{1}{2}} + C|x - x'| \, \|q\| \, \|A^{(i)}(\cdot, y; \varkappa)\|,$$

which prove the asserted continuity of $A^{(i)}(x, y; \varkappa)$ $(i = 1, 2, 3, 4)$, while $A^{(0)}(x, y; \varkappa)$ is obviously continuous in x for $x \neq y$.

That $A^{(4)}(\cdot, y; \varkappa) \in B$ follows from (2.11) and the remark after Lemma 4.1.

The above facts together with Lemmas 4.3 and 4.5 show that a unique solution $H^{(4)}(x, y; \varkappa)$ of the modified kernel equation exists for fixed y. If we prove that $\|H^{(4)}(\cdot, y; \varkappa)\| \leqq C$, where C depends only on $\varkappa$, we can complete the proof of the theorem in view of Theorem 1, ii).

From (1.4) we see that $H^{(4)}(x, y; \varkappa)$ is the sum of $A^{(4)}(x, y; \varkappa)$ and $f(x) = -R_{0\varkappa^2} V H^{(4)}(x, y; \varkappa)$ for fixed y. It is easily seen that $\|f\| \leqq \|R_{0\varkappa^2}\| \, \|q\| \, \|H^{(4)}(\cdot, y; \varkappa)\|_B$ is independent of y, for we have

$$\|H^{(4)}(\cdot, y; \varkappa)\|_B \leqq \|(I - T_\varkappa)^{-1}\|_B \, \|A^{(4)}(\cdot, y; \varkappa)\|_B.$$

Now in virtue of (2.12) we have proved the desired result.

II. Eigenfunction expansions

§ 6. Existence of the eigenfunctions. In quantum-mechanical problems of potential scattering, the *distorted plane wave* plays an important role. It is generally assumed to be a bounded solution of the Schroedinger equation

$$(6.1) \qquad -\Delta \varphi(x) + q(x)\, \varphi(x) = |k|^2 \varphi(x)$$

with the asymptotic conditions $v(x) = \varphi(x) - e^{ik\cdot x} = O(|x|^{-1})$ and $|x| \left(\partial v(x)/\partial |x| - i|k| v(x)\right) \to 0$ (radiation condition) for $|x| \to \infty$. Here k is a 3-dimensional vector called the *wave vector* and $|k|$ denotes its length. We shall denote by M the totality of the wave vectors k, which is, of course, isomorphic to E. $\varphi(x)$ has the form (plane wave) + (outgoing wave). It is also possible to consider the distorted plane wave which has the form (plane wave) + (incoming wave) in quite a similar way. As is described in most of the physical literature [23], the distorted plane wave $\varphi(x)$ is obtained as a solution of the integral equation

$$(6.2) \qquad \varphi(x, k) = e^{ik\cdot x} - \frac{1}{4\pi} \int\limits_E \frac{e^{i|k|\,|x-y|}}{|x-y|} q(y)\, \varphi(y, k)\, dy,$$

which is more convenient for mathematically rigorous treatment than (6.1), for (6.1) is not valid at singularities of $q(x)$. Let us introduce the function

$$(6.3) \qquad p(x, k) = -\frac{1}{4\pi} \int\limits_E \frac{e^{i|k|\,|x-y|}}{|x-y|} q(y)\, e^{ik\cdot y} dy.$$

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[23] See *e.g.* MOTT & MASSEY [*23*].

Then, if we put $v(x, k) = \varphi(x, k) - e^{ik\cdot x}$, (6.2) becomes

$$v(x, k) = p(x, k) - \frac{1}{4\pi} \int_E \frac{e^{i|k||x-y|}}{|x-y|} q(y)\, v(y, k)\, dy. \tag{6.4}$$

Since $p(\cdot, k) \in B$ as is easily seen from the remark after Lemma 4.1, (6.4) has a unique solution $v(\cdot, k) \in B$ for each fixed k ($|k| > 0$) in virtue of Lemmas 4.3 and 4.4. Moreover, we see from Lemma 4.1 and the remark thereafter that $v(x, k)$ as well as $p(x, k)$ is Hoelder continuous in x. Since $q(x)$ is Hoelder continuous, $\varphi(x, k)$ is then seen to satisfy (6.1). Thus we arrive at the following

Theorem 3. *There exists a unique solution $v(x, k) \in B$ of (6.4) for $|k| > 0$ and $\varphi(x, k) = e^{ik\cdot x} + v(x, k)$ is a solution of (6.1) and (6.2). Moreover, $\varphi(x, k)$ as well as $v(x, k)$ is bounded and uniformly continuous in x and k for $x \in E$ and $k \in D$, where D is any compact domain of M not containing the origin.*

$\varphi(x, k)$ is called an *eigenfunction* associated with the *eigennumber* $|k|^2$, and we can speak of an ∞^3-continuous family of eigenfunctions in conformity with various magnitudes and directions of the wave vectors k. However, we should remark here that under our assumptions (A) on $q(x)$, $v(x, k)$ does not always satisfy the asymptotic conditions mentioned above [24].

Proof of Theorem 3. We have only to prove the last statement. From the remark after Lemma 4.1 it follows that $|p(x, k)| \leqq C$, where C is independent of x and k. Since (6.4) can be written as $v(\cdot, k) = (I - T_{|k|})^{-1} p(\cdot, k)$, we see from Lemma 4.7 that $v(x, k)$ is bounded in $x \in E$ and $k \in D$. Similarly it follows from Lemma 4.8 that $v(x, k)$ is uniformly continuous in x and k in the same domain, if we show that $p(\cdot, k)$ is strongly continuous in $k \in D$.

Given any $\delta > 0$, we have for a sufficiently large $R = R(\delta)$ and for $|x| \geqq R$

$$|p(x, k)| \leqq C \int_E \frac{|q(y)|}{|x-y|} dy < \delta/2 \tag{6.5}$$

by Lemma 3.1. For $|x| < R$ we have

$$\begin{aligned} |p(x, k_1) - p(x, k_2)| \leqq{} & C \int_E \frac{|e^{i|k_1||x-y|} - e^{i|k_2||x-y|}|}{|x-y|} |q(y)|\, dy + \\ & + C \int_E \frac{|q(y)|}{|x-y|} |e^{ik_1\cdot y} - e^{ik_2\cdot y}|\, dy \\ \leqq{} & C \int_{K(R_1)} ||k_1| - |k_2||\, |q(y)|\, dy + C \int_{K(R_1)'} \frac{|q(y)|}{|x-y|} dy + \\ & + C \int_{K(R_1)} |k_1 - k_2|\, |y| \frac{|q(y)|}{|x-y|} dy + C \int_{K(R_1)'} \frac{|q(y)|}{|x-y|} dy \\ ={} & J_1 + J_2 + J_3 + J_4. \end{aligned}$$

Here we can choose an $R_1 = R_1(\delta)$ sufficiently large so that each of J_2 and J_4 may be bounded by $\delta/4$ (see the estimation of J_2' in the proof of Lemma 2.2). On the other hand, J_1 and J_3 can be made arbitrarily small ($J_1 < \delta/4$, $J_3 < \delta/4$)

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[24] These conditions are satisfied if we assume that $q(x) = O(|x|^{-3-h})$ ($|x| \to \infty$).

by taking $|k_1-k_2|\geqq||k_1|-|k_2||$ small. Together with (6.5) we see that

$$|p(x,k_1)-p(x,k_2)|<\delta \qquad (x\in E,\ k_1,\ k_2\in M)$$

if $|k_1-k_2|$ is sufficiently small. This proves the desired continuity of $p(\cdot,k)$ and completes the proof of Theorem 3.

§ 7. The spectrum of *H*. It is generally believed that the positive real line is occupied by what are called the *continuous eigenvalues* of H which are, in our terminology, *eigennumbers*, while in the negative half there exist possibly discrete eigenvalues but no continuous spectrum. This is certainly the case under our assumptions (A) on $q(x)$. But from the mathematical standpoint the non-existence of positive eigenvalues is not self-evident. This problem has been raised by Povzner [*28*], though he only commented on its plausibility and proved it for the case where $q(x)$ is a function of $|x|$ at least for large $|x|$. Recently, however, Kato [*18*] has proved a theorem[25] which implies the non-existence of positive eigenvalues in our case. We shall give a proof of this fact later (Theorem 6). Concerning the features of the negative part of the spectrum of H, we have

Theorem 4. *On the negative real line the continuous spectrum of H is absent*[26]. *The negative eigenvalues, if they exist, are of finite multiplicity and are discrete in the sense that they form an isolated set having no limit point other than the origin* 0.

Proof. Let a be sufficiently large so that we may have $H+a=H_0+V+a>a'>0$. (Note that H is bounded from below; see § 1.) Since the spectrum of H_0 occupies the full interval $[0,\infty)$, that of $(H_0+a)^{-1}$ is the interval $[0,1/a]$. Now the second resolvent equation gives $(H+a)^{-1}=(H_0+a)^{-1}-(H+a)^{-1}V(H_0+a)^{-1}$, where $(H+a)^{-1}V(H_0+a)^{-1}$ is a completely continuous operator, for $-a$ lies in the resolvent sets of both H and H_0 and, as we have seen in § 2, $V(H_0+a)^{-1}$ is of the Hilbert-Schmidt type. According to a well-known theorem of Weyl[27] that asserts the invariance of the set of limit points[28] of the spectrum of a self-adjoint operator by the addition of a completely continuous operator, the spectrum of $(H+a)^{-1}$ consists of the whole interval $[0,1/a]$ and, possibly, of isolated points outside $[0,1/a]$ (which may have limit point $1/a$). This means that, on the negative real axis, H may have only discrete eigenvalues of finite multiplicity having, possibly, limit point at 0. This completes the proof of Theorem 6.

Hereafter we shall denote by μ_n the discrete negative eigenvalues counted according to their multiplicity and by φ_n the orthonormalized *eigenvectors* (see footnote 3) associated with μ_n.

§ 8. Expansion theorem. As we have pointed out in Introduction, we can use the distorted plane waves $\varphi(x,k)$ constructed in § 6 as the eigenfunctions for expanding an arbitrary function.

Theorem 5. *Let $f(x)$ be an arbitrary L_2-function.* i) *Then the generalized Fourier transform*

$$\hat f(k)=(2\pi)^{-\frac{3}{2}}\,\mathrm{l.i.m.}\int\limits_E \overline{\varphi(x,k)}\,f(x)\,dx \tag{8.1}$$

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[25] *Cf.* Lemma 4.4.

[26] *Cf.* also Povzner [*28*], Chapter II, Theorem 6.

[27] See Riesz & Nagy [*29*], p. 367.

[28] Eigenvalues of infinite multiplicity are included among the "limit points".

of $f(x)$ *exists and belongs to* $L_2(M)$, *where* $\text{l.i.m.}\int_E \ldots dx$ *means the limit in the mean of the function* $\int_{K(N)} \ldots dx$ *as* $N \to \infty$. *Also the generalized Fourier coefficient*

$$\hat{f}_n = \int_E \overline{\varphi_n(x)}\, f(x)\, dx = (f, \varphi_n) \tag{8.2}$$

of $f(x)$ *exists, and* $\sum_{n=1}^{\infty} |\hat{f}_n|^2 < \infty$.

ii) *We have the following expansion formulas:*

$$f(x) = (2\pi)^{-\frac{3}{2}}\, \text{l.i.m.} \int_M \varphi(x, k)\, f(k)\, dk + \underset{N \to \infty}{\text{l.i.m.}} \sum_{n=1}^{N} f_n\, \varphi_n(x)\,, \tag{8.3}$$

$$\|f\|^2 = \int_M |\hat{f}(k)|^2\, dk + \sum_{n=1}^{\infty} |\hat{f}_n|^2 \qquad (\text{PARSEVAL'S } \textit{equality})\,, \tag{8.4}$$

$$(f, g) = \int_M \hat{f}(k)\, \overline{\hat{g}(k)}\, dk + \sum_{n=1}^{\infty} \hat{f}_n\, \overline{\hat{g}_n} \qquad (\textit{generalized } \text{PARSEVAL'S } \textit{equality})\,. \tag{8.5}$$

In (8.5) $g \in L_2(E)$, *and in* (8.3) $\text{l.i.m.}\int_M \ldots dk$ *means* $\underset{N\to\infty,\, \varepsilon\to\infty}{\text{l.i.m.}} \int_{K(N)-K(\varepsilon)} \ldots dk$.

iii) *Conversely let* $F(k) \in L_2(M)$ *be given. Then we can construct an* L_2*-function* $f(x)$ *through formula* (8.3) *with* $\hat{f}(k) = F(k)$, $\hat{f}_n = 0$ *such that* (8.1), (8.2), (8.4) *and* (8.5) *hold good.*

iv) *Let* E_μ *be the resolution of the identity associated with* H, *and let* $P = I - E_0$. *Then the part in* $PL_2(E)$ *of* H *is unitarily equivalent to* H_0.

v) $f \in D(H) = D(H_0)$ *if and only if* $|k|^2 \hat{f}(K) \in L_2(M)$ *and* $\sum_{n=1}^{\infty} \mu_n^2 |\hat{f}_n|^2 < \infty$. *We have the following representation of* H:

$$Hf(x) = (2\pi)^{-\frac{3}{2}}\, \text{l.i.m.} \int_M |k|^2 \varphi(x, k)\, \hat{f}(k)\, dk + \underset{N \to \infty}{\text{l.i.m.}} \sum_{n=1}^{N} \mu_n\, \hat{f}_n\, \varphi_n(x)\,. \tag{8.6}$$

The proof of i), ii) and v) except for the complete domain characterization of H will be given in the following section; it seems appropriate to remark here that the proof depends on some properties of the spectrum of H (Theorem 4) but not on the absolute continuity of the positive part of the spectrum (Theorem 6), which will be established in the course of the proof of ii). We shall also make use of Lemma 4.4, which is *almost* equivalent to the non-existence of the positive eigenvalues of H, but not in the strict sense, for we have not shown that every eigenvector associated with any positive eigenvalues should satisfy the homogeneous integral equation (4.7) and that it should be a B-function. Pertaining to the positive part of the spectrum of H, we have

Theorem 6. *There exists no positive eigenvalue of* H. *Moreover, the spectrum on the positive real line is absolutely continuous.*

The proof of Theorem 5, iii), iv) and the domain characterization of H will be given in connection with the proof, by the aid of the time-dependent theory, that the S-matrix is unitary.

It is implied by (8.6) that the Schroedinger operator $H = -\Delta + V$ admits of a *diagonal representation*, or H is *diagonalizable*, in terms of the eigenvectors $\varphi_n(x)$ and the eigenfunctions $\varphi(x, k)$.

§ 9. Proof of the completeness of $\{\varphi_n(x)\}$ and $\{\varphi(x, k)\}$. We shall divide the discussions that follow into two parts. One concerns the projection P and the other E_0. So to speak, we shall first take up the expansion problem for the function $Pf(x)$ and then extend our research to the function $E_0 f(x)$.

Let us consider the conjugate Fourier transform $g(x, k; \varkappa)$ of $H(x, y; \varkappa)$ with the defining equation

$$(9.1) \qquad g(x, k; \varkappa) = (2\pi)^{-\frac{3}{2}} \int_E H(x, y; \varkappa)\, e^{ik\cdot y}\, dy$$

for each x and $\operatorname{Im}\varkappa > 0$, $\operatorname{Im}\varkappa^2 \neq 0$, noting that $H(x, \cdot; \varkappa) \in L_2(E)$ (see Theorem 2 and Lemma 2.1). Actually, however, "l.i.m." in the above definition is unnecessary. This will be made clear by the following

Lemma 9.1. *If* $\operatorname{Im}\varkappa > 0$ *and* $\operatorname{Im}\varkappa^2 \neq 0$, *then* $H(x, y; \varkappa)$ *is absolutely integrable in* x *or* y *and* $g(x, k; \varkappa)$ *is a bounded function of* x *and* k *for* $x \in E$ *and* $k \in M$.

Proof. According to the kernel equation (1.1) for $H(x, y; \varkappa) = G(x, y; \varkappa^2)$ we have $(\varkappa = a + ib)$

$$\int_E |H(x, y; \varkappa)|\, dx \leqq C \int_E \frac{e^{-b|x-y|}}{|x-y|}\, dx + C \int_E |q(z)\, H(z, y; \varkappa)|\, dz \int_E \frac{e^{-b|x-z|}}{|x-z|}\, dx.$$

A further estimation is possible by (1.5) and SCHWARZ's inequality, which gives

$$\int_E |H(x, y; \varkappa)|\, dx \leqq C + C\, \|q\|\, \|H(\cdot, y; \varkappa)\| \leqq C,$$

where C depends on $\varkappa$ but not on y and is finite if $\operatorname{Im}\varkappa > 0$ and $\operatorname{Im}\varkappa^2 \neq 0$ (see Theorem 2). By the symmetry stated in Lemma 2.1 $H(x, y; \varkappa)$ is absolutely integrable in y, too, and hence it follows from (9.1) that $g(x, k; \varkappa)$ is bounded in x and k for $x \in E$ and $k \in M$.

Set

$$(9.1)' \qquad h(x, k; \varkappa) = (2\pi)^{\frac{3}{2}} (|k|^2 - \varkappa^2)\, g(x, k; \varkappa) = e^{(ik\cdot x)} + h^{(1)}(x, k; \varkappa).$$

Then we have

Lemma 9.2. $h^{(1)}(x, k; \varkappa)$ *is a B-function of* x *for* $k \in M$, $\operatorname{Im}\varkappa > 0$ *and* $\operatorname{Im}\varkappa^2 \neq 0$ *and satisfies the integral equation*

$$(9.2) \qquad h^{(1)}(x, k; \varkappa) = p(x, k; \varkappa) - \frac{1}{4\pi} \int_E \frac{e^{i\varkappa|x-y|}}{|x-y|}\, q(y)\, h^{(1)}(y, k; \varkappa)\, dy,$$

where

$$(9.3) \qquad p(x, k; \varkappa) = -\frac{1}{4\pi} \int_E \frac{e^{i\varkappa|x-y|}}{|x-y|}\, q(y)\, e^{ik\cdot y}\, dy$$

is a B-function of x *for* $k \in M$ *and* $\operatorname{Im}\varkappa \geqq 0$. *Moreover, we can extend the definition of* $h(x, k; \varkappa)$ *to the case where* $\operatorname{Im}\varkappa = 0$ *and* $\operatorname{Re}\varkappa > 0$. $h(x, k; \varkappa)$ *is bounded and uniformly continuous in* x, k *and* $\varkappa$ *for* $x \in E$, $k \in M$ *and* $\operatorname{Im}\varkappa \geqq 0$, $0 < \alpha \leqq \operatorname{Re}\varkappa \leqq \beta < \infty$. *In particular,* $h(x, k; |k|) = \varphi(x, k)$.

Proof. Let us first show that $g(x, k; \varkappa)$ satisfies the equation

$$g(x, k; \varkappa) = (2\pi)^{-\frac{3}{2}} \frac{e^{ik\cdot x}}{|k|^2 - \varkappa^2} - \frac{1}{4\pi} \int_E \frac{e^{i\varkappa|x-y|}}{|x-y|} q(y)\, g(y, k; \varkappa)\, dy. \tag{9.4}$$

From the kernel equation (1.1) we have for any $f \in L_2(E)$

$$\int_E H(x, y; \varkappa) f(y)\, dy = \frac{1}{4\pi} \int_E \frac{e^{i\varkappa|x-y|}}{|x-y|} f(y)\, dy -$$

$$- \frac{1}{4\pi} \int_E \frac{e^{i\varkappa|x-z|}}{|x-z|} q(z)\, dz \int_E H(z, y; \varkappa) f(y)\, dy,$$

where we have performed interchange of the order of integrations, which is permitted since the last integral is absolutely convergent because $\int_E |\;|\, dz \int_E |\;|\, dy \leqq C \|q\|\, \|H(z, \cdot; \varkappa)\|\, \|f\| \leqq C \|q\|\, \|f\|$ (by Theorem 2). Introducing the Fourier transform of $f(x)$ by

$$\hat{f}_0(k) = (2\pi)^{-\frac{3}{2}} \operatorname{l.i.m.} \int_E e^{-ik\cdot x} f(x)\, dx \tag{9.5}$$

and making use of PARSEVAL's equality, we get therefore

$$\begin{aligned} \int_M g(x, k; \varkappa) \hat{f}_0(k)\, dk &= (2\pi)^{-\frac{3}{2}} \int_M \frac{e^{ik\cdot x}}{|k|^2 - \varkappa^2} \hat{f}_0(k)\, dk - \\ &- \frac{1}{4\pi} \int_M \left[\int_E \frac{e^{i\varkappa|x-z|}}{|x-z|} q(z)\, g(z, k; \varkappa)\, dz \right] \hat{f}_0(k)\, dk \end{aligned} \tag{9.6}$$

where we have used the fact that the conjugate Fourier transform of $(4\pi|x-y|)^{-1} \times e^{i\varkappa|x-y|}$ as function of y is $(2\pi)^{-\frac{3}{2}} (|k|^2 - \varkappa^2)^{-1} e^{ik\cdot x}$ and have interchanged the order of integrations in the last integral, noting that $|x-z|^{-1} e^{i\varkappa|x-z|} q(z) \in L_1$ and that $\int_M |g(z, k; \varkappa)|\, |\hat{f}_0(k)|\, dk \leqq C \|H(z, \cdot; \varkappa)\|\, \|f\| \leqq C \|f\|$ (by Theorem 2). Since $\hat{f}_0 \in L_2(M)$ is arbitrary, (9.6) yields (9.4). Then the equation (9.2) for $h^{(1)}(x, k; \varkappa)$ follows at once from (9.1)′ and (9.4).

From the remark after Lemma 4.1 we can see that $p(\cdot, k; \varkappa) \in B$ and that $p(x, k; \varkappa)$ is bounded for $x \in E$, $k \in M$ and $\operatorname{Im} \varkappa \geqq 0$. Similarly, the second term on the right of (9.2) is also in B if $\operatorname{Im} \varkappa > 0$ and $\operatorname{Im} \varkappa^2 \neq 0$ (see Lemma 9.1). Thus $h^{(1)}(\cdot, k; \varkappa) \in B$ for $\operatorname{Im} \varkappa > 0$, $\operatorname{Im} \varkappa^2 \neq 0$, so that we can rewrite (9.2) as $h^{(1)}(\cdot, k; \varkappa) = (I - T_\varkappa)^{-1} p(\cdot, k; \varkappa)$. Noting that $p(\cdot, k; |k|) = p(\cdot, k) \in B$, this proves the lemma in view of Lemma 4.8 and Theorem 3, if we show that $p(\cdot, k; \varkappa) \in B$ is strongly continuous uniformly in $k \in M$ and $\varkappa$ for $\operatorname{Im} \varkappa \geqq 0$. The uniform continuity in k can be proved by an argument similar to, but simpler than, the one used for proving the uniform continuity in x and k of $\varphi(x, k)$ in § 6. Thus it remains to prove the uniform continuity in $\varkappa$ of $p(\cdot, k; \varkappa)$.

Given any $\delta > 0$, we can choose an $R = R(\delta)$ so that for $|x| \geqq R$ we have

$$|p(x, k; \varkappa)| \leqq C \int_E \frac{|q(y)|}{|x-y|} dy < \delta/2 \tag{9.7}$$

y virtue of Lemma 3.1. For $|x|<R$

$$|p(x,k;\varkappa_1)-p(x,k;\varkappa_2)|\leqq C\int_E \frac{|e^{i\varkappa_1|x-y|}-e^{i\varkappa_2|x-y|}|}{|x-y|}|q(y)|\,dy$$

$$\leqq C\int_{K(R_1)} \frac{|e^{i\varkappa_1|x-y|}-e^{i\varkappa_2|x-y|}|}{|x-y|}|q(y)|\,dy+C\int_{K(R_1)'}\frac{|q(y)|}{|x-y|}\,dy$$

$$=J+J'.$$

Again we can choose an $R_1=R_1(\delta)$ so large that J' may be bounded by $\delta/2$ (see the estimation of J_2' in the proof of Lemma 2.2). In order to estimate J, we first note that

$$|e^{i\varkappa_1|x-y|}-e^{i\varkappa_2|x-y|}|=\Big|i|x-y|\int_L e^{i\nu|x-y|}d\nu\Big|\leqq|\varkappa_1-\varkappa_2|\,|x-y|,$$

where L denotes a straight line which starts from $\varkappa_2$ and ends in $\varkappa_1$. Application of this inequality to J gives

$$J\leqq C|\varkappa_1-\varkappa_2|\int_{K(R_1)}|q(y)|\,dy.$$

Taking $\varkappa_1$ and $\varkappa_2$ so near to each other that $J<\delta/2$, we are led, noting (9.7) and the above estimates for J and J', to the result that for any given $\delta>0$ we have

$$|p(x,k;\varkappa_1)-p(x,k;\varkappa_2)|<\delta \qquad (x\in E,\ k\in M), \tag{9.8}$$

if $|\varkappa_1-\varkappa_2|$ is sufficiently small. (9.8) proves the required uniform continuity in $\varkappa$ of $p(\cdot,k;\varkappa)$.

Now we shall enter into the expansion problem for $(Pf)(x)$, where we first assume that $f(x)\in C_0^\infty(E)$. As the first step we prove Parseval's equality.

Since $g(x,k;\varkappa)$ is the conjugate Fourier transform of $H(x,y;\varkappa)$, it follows from Parseval's equality that for $\operatorname{Im}\varkappa>0$ and $\operatorname{Im}\varkappa^2\neq 0$

$$\begin{aligned}\int_E H(z,x;\varkappa)\,\overline{H(z,y;\varkappa)}\,dz&=\int_M g(x,k;\varkappa)\,\overline{g(y,k;\varkappa)}\,dk\\ &=\frac{1}{(2\pi)^3(\varkappa^2-\overline{\varkappa^2})}\int_M\left[\frac{1}{|k|^2-\varkappa^2}-\frac{1}{|k|^2-\overline{\varkappa^2}}\right]\times\\ &\qquad\times h(x,k;\varkappa)\,\overline{h(y,k;\varkappa)}\,dk.\end{aligned}\tag{9.9}$$

Let us multiply both sides of (9.9) by $\overline{f(x)}$ and $f(y)$ and integrate over E with respect to x and y. Then the left-hand side gives, on multiplying by $(\varkappa^2-\overline{\varkappa^2})$,

$$(\varkappa^2-\overline{\varkappa^2})\,(R_{\overline{\varkappa^2}}f,R_{\overline{\varkappa^2}}f)=(\varkappa^2-\overline{\varkappa^2})\,(R_{\varkappa^2}R_{\overline{\varkappa^2}}f,f)=\big((R_{\varkappa^2}-R_{\overline{\varkappa^2}})f,f\big),\tag{9.10}$$

where we have made use of the first resolvent equation[29] $R_\lambda-R_\varkappa=(\lambda-\lambda')R_\lambda R_{\lambda'}$ and freely interchanged the order of integrations in the integral considered, noting that it is absolutely convergent:

$$\int_E\int_E\|H(\cdot,x;\varkappa)\|\,\|H(\cdot,y;\varkappa)\|\,|f(x)|\,|f(y)|\,dy\,dx<\infty.$$

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[29] See Hille & Phillips [*12*], p. 126.

Here let us introduce the function

$$\Phi(k;\varkappa)=(2\pi)^{-\frac{3}{2}}\int_E \overline{h(x,k;\varkappa)}\,f(x)\,dx \qquad \bigl(f(x)\in C_0^\infty(E)\bigr). \tag{9.11}$$

Before proceeding further we note that $\Phi(k;\sqrt{\mu+i\varepsilon})$ is bounded for $k\in M$, $\mu\in[\alpha,\beta]$[30] $(0<\alpha<\beta)$ and $0\leqq\varepsilon\leqq\varepsilon_0$ $(\varepsilon_0>0)$, as is easily seen from Lemma 9.2 and that $\Phi(k,\varkappa)\to\hat{f}(k)$ as $\varkappa\to|k|$, as is seen from (8.1) and the fact that $f(x)\in C_0^\infty(E)$, and hence the integral is well defined.

Let us put $\varkappa^2=\mu+i\varepsilon$. Since $f(x)\in C_0^\infty(E)$ and the integrand of the right side of (9.9)

$$\frac{2i\varepsilon}{(|k|^2-\mu)^2+\varepsilon^2}\,h(x,k;\sqrt{\mu+i\varepsilon})\,\overline{h(y,k;\sqrt{\mu+i\varepsilon})}$$

is absolutely integrable in k and bounded in x and y, we can freely interchange the order of integration in forming the above-mentioned multiple integral on the right-hand side of (9.9). This enables us to obtain in view of (9.10), for $\varepsilon>0$,

$$\bigl((R_{\mu+i\varepsilon}-R_{\mu-i\varepsilon})f,f\bigr)=\int_M\frac{2i\varepsilon}{(|k|^2-\mu)^2+\varepsilon^2}\,|\Phi(k;\sqrt{\mu+i\varepsilon})|^2\,dk. \tag{9.12}$$

Now we can avail ourselves of the fundamental relation

$$\frac{1}{2}\bigl[((E_\beta+E_{\beta-0})f,f)-((E_\alpha+E_{\alpha-0})f,f)\bigr]=\frac{1}{2\pi i}\lim_{\varepsilon\downarrow 0}\int_\alpha^\beta\bigl((R_{\mu+i\varepsilon}-R_{\mu-i\varepsilon})f,f\bigr)\,d\mu\,{}^{31},$$

which, incorporated with (9.12), yields for α and β such that $0<\alpha<\beta$

$$\begin{aligned}&\frac{1}{2}\bigl[((E_\beta+E_{\beta-0})f,f)-((E_\alpha+E_{\alpha-0})f,f)\bigr]\\&\qquad=\frac{1}{\pi}\lim_{\varepsilon\downarrow 0}\int_\alpha^\beta d\mu\int_M\frac{\varepsilon}{(|k|^2-\mu)^2+\varepsilon^2}\,|\Phi(k;\sqrt{\mu+i\varepsilon})|^2\,dk\\&\qquad=\frac{1}{\pi}\lim_{\varepsilon\downarrow 0}\int_M dk\int_\alpha^\beta\frac{\varepsilon}{(|k|^2-\mu)^2+\varepsilon^2}\,|\Phi(k;\sqrt{\mu+i\varepsilon})|^2\,d\mu,\end{aligned}$$

where we have used the above-mentioned boundedness of $\Phi(k,\varkappa)$ and FUBINI's theorem. Let us consider the inner integral $\int_\alpha^\beta\ldots d\mu$. In view of the equation

$$\int_\alpha^\beta\frac{\varepsilon}{(|k|^2-\mu)^2+\varepsilon^2}\,d\mu=\tan^{-1}\frac{\beta-|k|^2}{\varepsilon}-\tan^{-1}\frac{\alpha-|k|^2}{\varepsilon},$$

we have for $\varepsilon\in[0,\varepsilon_0]$ $(\varepsilon_0>0)$

$$\begin{aligned}\left|\int_\alpha^\beta\ldots d\mu\right|\leqq F(k)&=C &&\text{if}\quad |k|\leqq\sqrt{\beta}+1,\\&=\frac{C}{(|k|^2-\beta)^2} &&\text{if}\quad |k|>\sqrt{\beta}+1,\end{aligned}$$

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[30] $[\alpha,\beta]$ denotes the closed interval $\alpha\leqq\mu\leqq\beta$.

[31] See *e.g.* STONE [*31*], p. 183.

where the C's do not depend on ε. $F(k)$ is integrable, and hence by the bounded convergence theorem we can interchange $\lim_{\varepsilon\downarrow 0}$ and the k-integration. Since $\Phi(k;\varkappa)$ tends to $\hat{f}(k)$ for $\varkappa\to|k|$ as has been seen above, we obtain for $f(x)\in C_0^\infty(E)$

$$\tfrac{1}{2}\left[((E_\beta+E_{\beta-0})f,f)-((E_\alpha+E_{\alpha-0})f,f)\right]=\int\limits_{\sqrt{\alpha}<|k|<\sqrt{\beta}}|\hat{f}(k)|^2\,dk; \tag{9.13}$$

here we have made use of the well-known relation[32]

$$\frac{1}{\pi}\lim_{\varepsilon\downarrow 0}\int_\alpha^\beta\frac{\varepsilon}{(a-\mu)^2+\varepsilon^2}f(\mu,\varepsilon)\,d\mu=\begin{cases}0 & \text{if}\quad a<\alpha\quad\text{or}\quad\beta<a\\ f(a,0) & \text{if}\quad\alpha<a<\beta,\end{cases}$$

in which $f(\mu,\varepsilon)$ is a continuous function of μ and ε for $\mu\in[\alpha,\beta]$ and $\varepsilon\in[0,\varepsilon_0]$ $(\varepsilon_0>0)$. Letting $\alpha\to\beta$ in (9.13), we see that $\left((E_\beta-E_{\beta-0})f,f\right)=0$ for any $f(x)\in C_0^\infty(E)$, which is dense in $L_2(E)$ in the L_2-norm. This shows that $E_\beta=E_{\beta-0}$ and β is not an eigenvalue of H, and since β is positive and otherwise arbitrary, we conclude that no positive eigenvalues exist. This proves the first half of Theorem 6. Hence we have from (9.13)

$$((E_\beta-E_\alpha)f,f)=\int\limits_{\sqrt{\alpha}<|k|<\sqrt{\beta}}|\hat{f}(k)|^2\,dk \tag{9.14}$$

for $f(x)\in C_0^\infty(E)$. Letting $\alpha\to 0$ and $\beta\to\infty$, we arrive at PARSEVAL's equality

$$\|Pf\|^2=\int\limits_M|\hat{f}(k)|^2\,dk \tag{9.15}$$

for $f(x)\in C_0^\infty(E)$. We have so far assumed that $f(x)\in C_0^\infty(E)$. But the extension to the general case $f(x)\in L_2(E)$ can be made in a standard manner[33]. Namely, we can show that $\hat{f}(k)$ exists and lies in $L_2(M)$ if we take the limit in the mean, as specified in Theorem 5, in forming $\hat{f}(k)$. Thus (9.14) and (9.15) hold true for $f(x)\in L_2(E)$.

Now let us step forward to the second problem, *i.e.*, the expansion of $(E_0f)(x)$. According to Theorem 4 the subspace $E_0L_2(E)$ is spanned by the eigenvectors $\{\varphi_n(x)\}$ belonging to the eigenvalues $\mu_n\leqq 0$, or $\{\varphi_n(x)\}$ form a complete orthonormal system in $E_0L_2(E)$. Consequently we have

$$\|E_0f\|^2=\sum_{n=1}^\infty|\hat{f}_n|^2. \tag{9.16}$$

In view of (9.15) and (9.16) we are led to the required relation

$$\|f\|^2=\|E_0f\|^2+\|Pf\|^2=\int\limits_M|\hat{f}(k)|^2\,dk+\sum_{n-1}^\infty|\hat{f}_n|^2. \tag{8.4}$$

Derivation of (8.5) from (8.4) is almost obvious[34].

Now let us prove (8.3). It is obvious from the completeness in $E_0\,L_2(E)$ of $\{\varphi_n(x)\}$ that

$$(E_0f)\,(x)=\underset{N\to\infty}{\text{l.i.m.}}\sum_{n=1}^N\hat{f}_n\,\varphi_n(x). \tag{9.17}$$

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[32] See *e.g.* TITCHMARSH [*33*], p. 31.

[33] See *e.g.* TITCHMARSH [*34*], pp. 55–56.

[34] See any books on Fourier integrals, *e.g.* TITCHMARSH [*33*], Chapter III.

It remains to show that

$$(9.18)\qquad (Pf)(x)=(2\pi)^{-\frac{3}{2}}\,\mathrm{l.i.m.}\int_M \varphi(x,k)\,\hat f(k)\,dk.$$

We can obtain from (9.15), if $g(x)\in C_0^\infty(E)$,

$$\begin{aligned}((E_\beta-E_\alpha)f,g)&=\int_{\sqrt{\alpha}<|k|<\sqrt{\beta}}\hat f(k)\,\overline{\hat g(k)}\,dk\\ &=\frac{1}{(2\pi)^{\frac{3}{2}}}\int_{\sqrt{\alpha}<|k|<\sqrt{\beta}}\hat f(k)\,dk\int_E\varphi(x,k)\,\overline{g(x)}\,dx\\ &=(2\pi)^{-\frac{3}{2}}\int_E\Big[\int_{\sqrt{\alpha}<|k|<\sqrt{\beta}}\varphi(x,k)\,\hat f(k)\,dk\Big]\overline{g(x)}\,dx,\end{aligned}$$

since $\varphi(x,k)$ is bounded and $\hat f(k)$ is integrable for $\sqrt{\alpha}<|k|<\sqrt{\beta}$ and $x\in E$. $C_0^\infty(E)$ being dense in $L_2(E)$ in the L_2-norm, we have

$$(9.19)\qquad ((E_\beta-E_\alpha)f)(x)=(2\pi)^{-\frac{3}{2}}\int_{\sqrt{\alpha}<|k|<\sqrt{\beta}}\varphi(x,k)\,\hat f(k)\,dk.$$

If we let $\alpha\to 0$ and $\beta\to\infty$, then the left side of (9.19) converges strongly to $(Pf)(x)$, and hence the limit in the mean of the right also exists and is equal to $(Pf)(x)$. This is nothing but the desired relation (9.18). We have proved i) and ii) of Theorem 5.

Now we show the diagonal representation (8.6) of H without complete characterization of the domain. For this purpose it suffices to show

$$(9.20)\qquad (Hf,g)=\int_M |k|^2\hat f(k)\,\overline{\hat g(k)}\,dk+\sum_{n=1}^{\infty}\mu_n\hat f_n\overline{\hat g}_n$$

for $f(x)\in D(H)=D(H_0)$ and $g(x)\in L_2(E)$. We see from (9.14) that

$$(9.21)\qquad \begin{aligned}(E_\mu f,g)&=\sum_{\mu_n\leqq\mu}\hat f_n\overline{\hat g}_n && (\mu<0)\\ &=\sum_{n=1}^{\infty}\hat f_n\overline{\hat g}_n+\int_{|k|<\sqrt{\mu}}\hat f(k)\,\overline{\hat g(k)}\,dk && (\mu\geqq 0);\end{aligned}$$

hence (9.20) holds if the relation

$$(Hf,g)=\int_{-\infty}^{\infty}\mu\,d(E_\mu f,g)$$

is taken into account.

Finally let us prove the last half of Theorem 6. Taking $g=f$ in (9.21), we can see that $\varrho_f(\mu)=(E_\mu f,f)$ is absolutely continuous in μ for $\mu>0$. This implies nothing but the absolute continuity of the spectrum of H on the positive real line.

III. S-matrices

§ 10. Wave operators $W_\pm$. As has been mentioned in the Introduction, we find it convenient to introduce the *wave operators* $W_\pm$ not only for the mathematical treatment of the S-matrix S, but also for the proof of Theorem 5, iii).

It is known[35] from the time-dependent theory of the S-matrices that, under the assumption that $q(x) \in L_2(E)$, the wave operators $W_\pm$ exist:

$$W_\pm = \underset{t \to \pm\infty}{s\text{-lim}}\, U(t)\ ^{36}, \qquad U(t) = e^{itH} e^{-itH_0}; \tag{10.1}$$

$W_\pm$ are isometric operators satisfying

$$PW_\pm = W_\pm, \tag{10.2}$$

which implies that the ranges of $W_\pm$ are subsets of $PL_2(E)$. The S-matrix is defined by

$$S = W_+^* W_-, \tag{10.3}$$

and it is generally believed to be unitary, which is, however, not self-evident. S is unitary if and only if the ranges of $W_\pm$ are identical. We shall prove that S is unitary by showing that the ranges of $W_\pm$ are identical and coincide with $PL_2(E)$.

To this end we introduce the operators $U_\pm$ by

$$(U_- f)(x) = (2\pi)^{-\frac{3}{2}}\, \text{l.i.m.} \int_M \varphi(x, k)\, \hat{f}_0(k)\, dk. \tag{10.4}$$

$$U_+ f = \overline{U_- \bar{f}}, \tag{10.5}$$

where $\hat{f}_0(k)$ is the Fourier transform of $f(x) \in L_2(E)$ (see (9.5)), and prove

Theorem 7. $U_\pm$, *formally defined by* (10.4) *and* (10.5), *are isometric operators with domains* $L_2(E)$ *and ranges* $PL_2(E)$, *and we have* $U_\pm = W_\pm$. S *defined by* (10.3) *is unitary.*

We shall prove the above theorem in the next section and then complete the proof of Theorem 5. Here we note that the statement iii) of Theorem 5 is equivalent to the following: Let Z be an isometric operator from $PL_2(E)$ to $L_2(M)$ defined by $Zf(k) = \hat{f}(k)$ $(f \in PL_2(E))$. Then Z is unitary. (The isometry of Z is already shown in (8.4).)

§ 11. Proof of Theorem 7. We first show that $U_\pm$ are well defined operators. To this end we need only to show that this is the case with U_-, since U_+ is related to U_- through (10.5).

Let us consider the integral

$$L_f(g) = \int_M \hat{f}_0(k)\, \overline{\hat{g}(k)}\, dk = (2\pi)^{-\frac{3}{2}} \int_M \hat{f}_0(k) \left[\int_E \varphi(x, k)\, \overline{g(x)}\, dx\right] dk, \tag{11.1}$$

where $g(x) \in C_0^\infty(E)$, $\hat{f}_0(k) \in C_0^\infty(M)$ and the carrier of $\hat{f}_0(k)$ does not contain the origin of M. The totality of such functions of k will be denoted by $C_0^\infty(M)'$. In (11.1) the x- and k-integrations are, actually, extended over compact domains and $\varphi(x, k)$ is bounded in k in a compact domain from which the origin is excluded (see Theorem 3). This fact allows us to interchange the order of integration, yielding

$$L_f(g) = (2\pi)^{-\frac{3}{2}} \int_E \left[\int_M \varphi(x, k)\, \hat{f}_0(k)\, dk\right] \overline{g(x)}\, dx. \tag{11.2}$$

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[35] See Cook [*7*], Kato [*17*] and Kuroda [*20*].

[36] "s-lim" denotes strong limit.

On the other hand it follows from (11.1) that

$$|L(g)| \leqq \Big(\int_M |\hat{f}_0(k)|^2\, dk \int_M |\hat{g}(k)|^2\, dk\Big)^{\frac{1}{2}} = \|f\|\,\|Pg\| \leqq \|f\|\,\|g\|.$$

Thus $L_f(g)$ is a densely defined, bounded linear functional of $g \in C_0^\infty(E)$. But such a functional can be extended to the whole $L_2(E)$, and we shall also denote the extension by $L_f(g)$. Then by means of RIESZ's theorem[37] there exists a unique element f^* of $L_2(E)$ such that $L_f(g) = (f^*, g)$, which defines an operator U by $Uf = f^*$ and $\|Uf\| \leqq \|f\|$. Now in view of (11.2), again restricting $g(x)$ to $C_0^\infty(E)$, we have

$$(Uf)(x) = (2\pi)^{-\frac{3}{2}} \int_M \varphi(x, k)\, \hat{f}_0(k)\, dk \quad a.e. \quad \big(f_0(k) \in C_0^\infty(M)\big). \tag{11.3}$$

A standard argument shows that (11.3) can be extended to every $f \in L_2(E)$ by writing $\mathrm{l.i.m.}\int_M \ldots dk$ instead of $\int \ldots dk$. Comparison of (11.3) with (10.4) gives the result that U is nothing but the required operator U_-. Thus we have proved that U_- is an everywhere defined, bounded operator $\big(D(U_-) = L_2(E)\big)$ and its norm $\|U_-\|$ does not exceed 1.

Next let us determine the adjoint U_-^* of U_-. This can be done most conveniently by using the relation $(U_-^* f, g) = (f, U_- g)$; the result is easily seen to be

$$(U_-^* f)(x) = (2\pi)^{-\frac{3}{2}}\, \mathrm{l.i.m.} \int_M e^{ik\cdot x} \hat{f}(k)\, dk. \tag{11.4}$$

In the derivation of (11.4) we have used reasoning quite analogous to that for getting (11.3).

U_- has the property that

$$U_- U_-^* = P, \tag{11.5}$$

which follows immediately from the defining equations (10.4), (11.4) of U_-, U_-^* and (8.3). Also we have

$$U_-^* H \subset H_0 U_-^*. \tag{11.6}$$

This can be seen from the following equations: For $f(x) \in D(H) = D(H_0)$

$$\begin{aligned}(U_-^* H f)(x) &= (2\pi)^{-\frac{3}{2}}\, \mathrm{l.i.m.} \int_M e^{ik\cdot x} (Hf)^\wedge(k)\, dk \\ &= (2\pi)^{-\frac{3}{2}}\, \mathrm{l.i.m.} \int_M e^{ik\cdot x} |k|^2 \hat{f}(k)\, dk; \\ (H_0 U_-^* f)(x) &= (2\pi)^{-\frac{3}{2}}\, \mathrm{l.i.m.} \int_M e^{ik\cdot x} |k|^2 (U_-^* f)\, \hat{f}_0(k)\, dk \\ &= (2\pi)^{-\frac{3}{2}}\, \mathrm{l.i.m.} \int_M e^{ik\cdot x} |k|^2 \hat{f}(k)\, dk,\end{aligned}$$

where we have made use of the diagonal representation of H (8.6) and the fact that H_0 is a special case of H with $V = 0$ and has no eigenvalues.

Now we can complete the proof of Theorem 7. Application of U_-^* from the left to both sides of (10.1) gives

$$U_-^* W_- = \lim_{t \to -\infty} U_-^* U(t) \qquad (\lim = s\text{-}\lim), \tag{11.7}$$

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[37] See *e.g.* RIESZ & NAGY [*29*] and YOSIDA [*39*].

since U_-^* is a bounded operator and $\lim_{t\to-\infty} U(t)$ exists. On the other hand, we see, by differentiating and again integrating, that

$$(U_-^* U(t) f, g) - (U_-^* f, g) = i \int_0^t (U_-^* e^{itH} V e^{-itH_0} f, g)\, dt \qquad (11.8)$$
$$- i \int_0^t (e^{itH_0} U_-^* V e^{-itH_0} f, g)\, dt,$$

where $f(x) \in D(H_0) = D(H)$ and $g(x) \in L_2(E)$ and we have made use of the fact that $U_-^* e^{itH} = e^{itH_0} U_-^*$, which is a consequence of (11.6). In view of (11.7) and (11.8), we obtain

$$(U_-^* W_- f, g) = (U_-^* f, g) + \lim_{t\to-\infty} i \int_0^t (e^{itH_0} U_-^* V e^{-itH_0} f, g)\, dt. \qquad (11.9)$$

The integrand of (11.9) is calculated as follows:

$$\begin{aligned}(e^{itH_0} U_-^* V e^{-itH_0} f, g) &= \int_M e^{it|k|^2} (U_-^* V e^{-itH_0} f)_0^{\wedge}(k)\, \overline{\hat{g}_0(k)}\, dk \\ &= \int_M e^{it|k|^2} (V e^{-itH_0} f)^{\wedge}(k)\, \overline{\hat{g}_0(k)}\, dk \\ &= \int_M (V e^{-it(H_0-|k|^2)} f)^{\wedge}(k)\, \overline{\hat{g}_0(k)}\, dk \\ &= (2\pi)^{-\frac{3}{2}} \int_M \left[\int_E \overline{\varphi(x,k)}\, q(x)\, (e^{-it(H_0-|k|^2)} f)(x)\, dx \right] \overline{\hat{g}_0(k)}\, dk,\end{aligned} \qquad (11.10)$$

where l.i.m. is not needed before $\int_M$, because $\varphi(x, k)$ is bounded in x, $q(x) \in L_2(E)$ and $(e^{-it(H_0-|k|^2)} f)(x) \in L_2(E)$.

Returning to (11.9), the limit for $t \to -\infty$ of the t-integral can be replaced by the Abelian limit, *i.e.*

$$\lim_{t\to-\infty} \int_0^t (e^{itH_0} U_-^* V e^{-itH_0} f, g)\, dt = \lim_{\varepsilon\downarrow 0} \int_0^{-\infty} e^{\varepsilon t} (e^{itH_0} U_-^* V e^{-itH_0} f, g)\, dt,$$

since the existence of the ordinary limit is known. Then (11.9) and (11.10) give

$$(U_-^* W_- f, g) = (U_-^* f, g) + \frac{i}{(2\pi)^{\frac{3}{2}}} \lim_{\varepsilon\downarrow 0} \int_0^{-\infty} e^{\varepsilon t}\, dt \,\times$$
$$\times \int_M \left[\int_E \overline{\varphi(x,k)}\, q(x)\, (e^{-it(H_0-|k|^2)} f)(x)\, dx \right] \overline{\hat{g}_0(k)}\, dk.$$

If we assume here that $\hat{g}_0(k) \in C_0^\infty(M)'$, then we can interchange the t- and k-integrations, for $\varphi(x, k)$ is bounded for $x \in E$ and $k \in D =$ carrier of $\hat{g}_0(k)$ (see Theorem 3) while

$$\int_E |q(x)|\, |e^{-it(H_0-|k|^2)} f(x)|\, dx \leqq \|q\|\, \|f\|.$$

$$(U_-^* W_- f, g) = (U_-^* f, g) + \frac{i}{(2\pi)} \lim_{\varepsilon\downarrow 0} \int_M \overline{\hat{g}_0(k)}\, dk \int_0^{-\infty} \left(e^{-it(H_0-(|k|^2-i\varepsilon))} f, F(\cdot, k)\right) dt.$$

where we have put

$$F(x, k) = \varphi(x, k)\, q(x) \in L_2(E) \qquad (\text{for each fixed } k \in M).$$

Hence

$$\begin{aligned}(U_-^* W_- f, g) &= (U_-^* f, g) - \frac{1}{(2\pi)^{\frac{3}{2}}} \lim_{\varepsilon \downarrow 0} \int_M \overline{\hat{g}_0(k)}\, dk \times \\ &\quad \times \lim_{t \to -\infty} \left([e^{-it(H_0 - (|k|^2 - i\varepsilon))} - 1]\, R_{0,\,|k|^2 - i\varepsilon} f, F(\cdot, k)\right), \\ &= (U_-^* f, g) + \frac{1}{(2\pi)^{\frac{3}{2}}} \lim_{\varepsilon \downarrow 0} \int_M \overline{\hat{g}_0(k)}\, \left(R_{0,\,|k|^2 - i\varepsilon} f, F(\cdot, k)\right) dk \\ &= (U_-^* f, g) + \frac{1}{(2\pi)^{\frac{3}{2}}} \lim_{\varepsilon \downarrow 0} \int_M \left(f, R_{0,\,|k|^2 + i\varepsilon}\, \varphi(\cdot, k)\, q(\cdot)\right) \overline{\hat{g}_0(k)}\, dk.\end{aligned} \tag{11.11}$$

Here we further assume that $f(x) \in C_0^\infty(E) \subset D(H_0)$. Now the function

$$\begin{aligned}\left(R_{0,\,|k|^2 + i\varepsilon}\, \varphi(\cdot, k)\, q(\cdot)\right)(x) &= \frac{1}{4\pi} \int_E \frac{e^{i\varkappa|x-y|}}{|x-y|}\, q(y)\, \varphi(y, k)\, dy \\ &= -\left(T_\varkappa \varphi(\cdot, k)\right)(x) \qquad (\varkappa = \sqrt{|k|^2 + i\varepsilon})\end{aligned}$$

converges to

$$\frac{1}{4\pi} \int_E \frac{e^{i|k|\,|x-y|}}{|x-y|}\, q(y)\, \varphi(y, k)\, dy = -\left(T_{|k|} \varphi(\cdot, k)\right)(x)$$

as $\varepsilon \downarrow 0$ uniformly with respect to $x \in E$ and k contained in the compact carrier of $\hat{g}_0(k)$ by virtue of Theorem 3 and Lemma 4.6. Thus the convergence of the integrand of (11.11) for $\varepsilon \downarrow 0$ is uniform, and we can interchange $\lim\limits_{\varepsilon \downarrow 0}$ and $\int\limits_M \ldots dk$ so as to obtain

$$\begin{aligned}\overline{(U_-^* W_- f, g)} &= \overline{(U_-^* f, g)} + \frac{1}{(2\pi)^{\frac{3}{2}}} \int_M \hat{g}_0(k)\, dk \int_E \overline{f(x)}\, dx\, \frac{1}{4\pi} \int_E \frac{e^{i|k|\,|x-y|}}{|x-y|}\, q(y)\, \varphi(y, k)\, dy \\ &= \frac{1}{(2\pi)^{\frac{3}{2}}} \int_M \hat{g}_0(k)\, dk \int_E \overline{f(x)} \left[\varphi(x, k) + \frac{1}{4\pi} \int_E \frac{e^{i|k|\,|x-y|}}{|x-y|}\, q(y)\, \varphi(y, k)\, dy\right] dx \\ &= \frac{1}{(2\pi)^{\frac{3}{2}}} \int_M \hat{g}_0(k)\, dk \int_E e^{ik\cdot x}\, \overline{f(x)}\, dx \qquad (\text{by } (6.2)) \\ &= \int_M \overline{\hat{f}_0(k)\, \overline{\hat{g}_0(k)}}\, dk = \overline{(f, g)}.\end{aligned} \tag{11.12}$$

$C_0^\infty(M)'$ is dense in $L_2(M)$, and hence the totality of such $g(x)$ that $\hat{g}_0(k) \in C_0^\infty(M)'$ is dense in $L_2(E)$. $C_0^\infty(E)$ is of course dense in $L_2(E)$. We can, therefore, obtain from (11.12) the following relation:

$$U_-^* W_- = I. \tag{11.13}$$

If we operate with U_- on both sides of (11.13) and take account of (10.2) and (11.5), we get

$$U_- = U_- U_-^* W_- = P W_- = W_- \tag{11.14}$$

and, on substituting this relation into (11.13),

$$U_-^* U_- = W^* W_- = I. \tag{11.15}$$

The relations (11.5) and (11.15) show that the operator U_- is an isometric operator with domain $D(U_-)=L_2(E)$ and range $R(U_-)=PL_2(E)$[38]. Quite a similar result follows for U_+ and W_+. Thus we can see that $S=W_+^* W_- = U_+^* U_-$ and S is unitary. Now the proof of Theorem 7 is complete.

In connection with eigenfunction expansion it remains to prove that the isometric operator Z is unitary, to characterize the domain of H, and to establish the unitary equivalence between the part of H in $PL_2(E)$ and H_0. To these ends we shall define the operators Y and Z' by

$$(Yf)(k) = \hat{f}_0(k) \qquad \left(\text{from } L_2(E) \ \textit{onto} \ L_2(M)\right);$$

$$(Z'f)(x) = (2\pi)^{-\frac{3}{2}} \operatorname{l.i.m.} \int_M \varphi(x,k)\, f(k)\, dk \qquad \left(\text{from } L_2(M) \text{ into } L_2(E)\right).$$

Then we have $U_- = Z'Y$ and $U_-^* = Y^{-1}Z$, and we have

$$I = U_-^* U_- = Y^{-1} Z Z' Y, \qquad Z Z' = Y Y^{-1} = I' \text{[39]}. \tag{11.16}$$

On the other hand (8.3), (8.4) and (8.5) imply that

$$P = Z'Z. \tag{11.17}$$

(11.16) and (11.17) show that Z is a unitary operator from $PL_2(E)$ onto $L_2(M)$, and that $Z'=Z^*$, the adjoint operator of Z (Z^* is a unitary operator from $L_2(M)$ onto $PL_2(E)$).

Now it follows from (8.6) that

$$(ZPHf)(k) = |k|^2 \hat{f}(k),$$

and it is known that H_0 represents the operation of multiplying by $|k|^2$ in $L_2(M)$. Thus we have proved iv) of Theorem 5.

Finally let us prove the first assertion of Theorem 5, v). Since the necessity is obvious from the diagonal representation (8.6) of H, it remains to show the sufficiency. We see from the unitarity of Z and (8.6) that if $F(k) \in L_2(M)$ and $|k|^2 F(k) \in L_2(M)$ then $Z^* F(x) \in PD(H) \subset D(H)$. Thus we need only to prove that for any sequence F_n such that $\sum |F_n|^2 < \infty$ and $\sum \mu_n^2 |F_n|^2 < \infty$, $\operatorname{l.i.m.} \sum F_n \varphi_n \in E_0 D(H) \subset D(H)$. But this follows easily from (8.6) and the fact that the correspondence between $E_0 L_2(E)$ and l_2[40], which assigns every $f \in E_0 L_2(E)$ to $\hat{f}_n \in l_2$, is one-to-one and isometric, where l_2 denotes the Hilbert space of all sequences $\{F_n\}$ such that $\sum |F_n|^2 < \infty$.

IV. Concluding remarks

§ 12. Higher-dimensional and two-dimensional cases. Our theory, which has so far been developed for the 3-dimensional Schroedinger operator, cannot be extended directly to the case of higher dimension. One of the main difficulties is that Theorems 1 and 2, which serve as a bridge between the resolvent R_λ and the kernel $G(x,y;\lambda)$, fail to hold, because the kernel $G(x,y;\lambda)$, even though it exists, is not of the Carleman type in the case of more than three dimensions.

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[38] $R(A)$ means the range of A.

[39] I' denotes the identity operator in $L_2(M)$.

[40] If the dimension m of $E_0 L_2(E)$ is finite then l_2 is the totality of finite sequences of length m.

A remedy for it has been suggested by GARDING [9]; it is to consider the operator $(H^t-\lambda)^{-1}$, where t is a certain positive integer, instead of $R_\lambda=(H-\lambda)^{-1}$. The operator $(H^t-\lambda)^{-1}$ is, in fact, known to have a kernel of the Carleman type. Of course, this will require many an alteration in the details of our theory.

On the contrary, in the 2-dimensional case our theory suffers no essential modification, for then the resolvent kernels can be shown to be of the Carleman type. But it should be remarked that results for the 2-dimensional case do not follow as special cases from the 3-dimensional treatment, because the potential function $q(x)$, which diminishes at infinity in the 2-dimensional Euclidian space E^2, does not always diminishes at infinity in $E=E^3$, which is an extension of E. The conditions imposed upon $q(x)$ are to be replaced by

(A′) *$q(x)$ is a real-valued function which is locally Hoelder continuous except at a finite number of singularities. Furthermore, $q(x)$ is square integrable $\left(q(x)\in L_2(E^2)\right)$ and behaves like $O(|x|^{-\frac{3}{2}-h})$ $(h>0)$ at infinity, i.e. there exist positive numbers h, C_0 and R_0 such that*

$$|q(x)|\leqq C_0|x|^{-\frac{3}{2}-h} \quad \textit{for} \quad |x|\geqq R_0.$$

Not all the details of the theory need be modified if we replace the function $(4\pi|x-y|)^{-1}e^{i\varkappa|x-y|}$, which is the resolvent kernel of $R_{0\varkappa^2}$, by[41] $\frac{1}{4}iH_0^{(1)}(\varkappa|x-y|)$, where $H_0^{(1)}$ denotes the Hankel function of first kind of order 0.

In concluding we remark that if we had assumed in addition to (A) that $q(x)\in L_1(E)$ or more explicitly $q(x)=O(|x|^{-3-h})$ at infinity (in the 3-dimensional case), our arguments could have been much simplified.

Acknowledgement. The writer wishes to express his sincere thanks to Professor T. KATO for his unceasing encouragement and valuable instruction.

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[41] See *e.g.* TITCHMARSH [35].

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(Received November 10, 1959)

Bounds in the Cauchy Problem for the Laplace Equation

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Communicated by C. LOEWNER

1. Introduction

It is well known that the Cauchy problem for the Laplace equation is not well posed in the sense of HADAMARD*, *i.e.*, the solution does not depend in a continuous way on the Cauchy data. Because of this fact and perhaps because of the feeling that such problems were unnatural, mathematicians have until recently neglected the study of the Cauchy problem for elliptic equations. However, during the past few years considerable attention has been given to rather general problems of this type, especially to the study of uniqueness of solution [*1*–*14*, *16*, *18*, *19*]**. One of the reasons for this recent interest is the fact that although such problems are not well posed, they nevertheless actually correspond to concrete physical situations. The physical problem is usually one in which a portion of the boundary is inaccessible for measurement of boundary data.

It has been demonstrated by LAURENTIEV [*12*] that if one prescribes in addition to the Cauchy data a supplementary condition, namely that the harmonic function be uniformly bounded by some constant M in its region of definition $\mathscr{D}$, then the Cauchy problem for the Laplace equation has a stable solution. The problem is then well posed. It will be noted in Section 2 that various other conditions are sufficient to insure stability.

In this paper we present a method for obtaining pointwise bounds for the solution to the Cauchy problem for the Laplace equation in n-dimensions. If the Cauchy data is known exactly the error term may be made small by the Rayleigh-Ritz technique. However, it usually happens that the Cauchy data is obtained by measurement and is not known precisely. On the other hand, an upper bound for the absolute value of the error in measurement of the data can usually be determined. We show that knowledge of this error and of an upper bound for the absolute value of the harmonic function in $\mathscr{D}$ leads to *a priori* bounds for the value of the harmonic function at any point in $\mathscr{D}$.

The method proposed here is in a certain sense a generalization of the work of LAURENTIEV [*12*]. In fact some of the results of LAURENTIEV may be used to give pointwise estimates near the portion of the boundary on which the data is given, but only if part of the data is homogeneous and if $\mathscr{D}$ satisfies certain geometrical requirements. Even if these conditions are met, Laurentiev's method still fails to yield pointwise estimates at every point in $\mathscr{D}$.

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* See, for instance, J. HADAMARD, Le Problème de Cauchy, Paris 1932, or reference [*17*].

** See also P. R. GARABEDIAN & LIEBERSTEIN, H. M., On the numerical calculation of detached bow shock waves in hypersonic flow. J. Aero. Sci. **25**, 109–118 (1958).

The results of this paper may be extended to more general types of equations or systems of equations, *e.g.*, polyharmonic equations, equations with variable coefficients, *etc.* We restrict consideration here to the Laplace equation, because it is the most important case physically, and because certain simplifications can be made which are not possible in the treatment of the more general problem. The more general problems will be dealt with in a subsequent paper.

2. Notation and Formulation of Problem

We consider an n-dimensional domain $\mathscr{D}$ bounded by a closed surface $\mathscr{C}$. On a portion Σ of $\mathscr{C}$ Cauchy data is prescribed. We denote the remainder of the surface $(\mathscr{C}-\Sigma)$ by $\bar{\Sigma}$, and take the origin of a spherical coordinate system at a point P_0. This point is chosen in such a way that there exists a sphere $K(r)$ of radius r and centre at P_0 which has the property that the intersection of $\mathscr{C}$ with $K(r)$ contains no points of $\bar{\Sigma}$. (If two or more disjoint regions are formed by the intersection of $K(r)$ and $\mathscr{D}$, it is sufficient that the boundary of any one of these portions contains no points of $\bar{\Sigma}$.) We denote by $\mathscr{D}_r$ the intersection of $K(r)$ and $\mathscr{D}$ (the portion whose boundary contains no points of $\bar{\Sigma}$ if two or more disjoint regions are formed), and require also that P_0 not be a point of $\mathscr{D}_r$. We denote by R the radius of the largest sphere satisfying the conditions prescribed above and by r_0 the radius of the smallest such sphere. We can always locate P_0 in such a way that $K(r_0)$ is tangent to $\mathscr{D}$ at some point of Σ. (One need only use a portion of Σ if necessary or convenient.) We assume this to be the case and prescribe further that P_0 be so chosen that $r_0>0$. (The point P_0 should not lie on Σ.) In most instances P_0 will not be a point of $\mathscr{D}$ but will lie outside of $\mathscr{C}$ near the portion Σ. It is obvious that we have considerable freedom in the choice of P_0. The optimum choice will depend on the geometry of Σ and upon the Cauchy data. Finally, we denote by $\omega(r)$ the portion of the surface of $K(r)$ which lies interior to $\mathscr{C}$.

Let w be a function uniformly bounded in $\mathscr{D}$ satisfying

$$\Delta w = 0, \qquad \Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} \tag{2.1}$$

therein and satisfying on the portion Σ of the boundary the conditions

$$\begin{aligned} &\int_{\Sigma} (w-f)^2\, d\sigma \leqq \alpha_1, \\ &\int_{\Sigma} (w_{,i}-F_i)\,(w_{,i}-F_i)\, d\sigma \leqq \alpha_2 \end{aligned} \tag{2.2}$$

where $d\sigma$ denotes an element of surface area on Σ, the comma denotes partial differentiation, and the repeated index denotes summation over i from 1 to n. The quantities f and F_i are the measured values of w and its gradient on Σ, and α_1, α_2 are known constants.

In order to obtain pointwise estimates for w at any point in $\mathscr{D}$ we first derive a suitable *a priori* estimate for an arbitrary function u which satisfies (2.1) in $\mathscr{D}$. Then by choosing u in an appropriate way we obtain the desired bounds for w.

We introduce the following notation:

$$\varepsilon_1 = \int_{\Sigma} u^2 \, d\sigma, \tag{2.3}$$

$$\varepsilon_2 = \int_{\Sigma} u_{,i} u_{,i} \, d\sigma$$

and

$$A_r(u) = \int_{\mathscr{D}_r} u_{,i} u_{,i} \, dx \tag{2.4}$$

where dx denotes an element of volume of $\mathscr{D}_r$. We assume further that

$$|u| \leqq M \tag{2.5}$$

in $\mathscr{D}$. We shall derive an *a priori* estimate for u of the following type:

$$|u(P)|^2 \leqq K_0 M^{2(1-\delta)} [K_1 \varepsilon_1 + K_2 \varepsilon_2]^{\delta} \tag{2.6}$$

where K_0, K_1, and K_2 are determined constants and δ is a known quantity $0 < \delta < 1$.

Before deriving (2.6) we point out how such an *a priori* estimate may be used to obtain pointwise estimates for the function w satisfying (2.1) and (2.2). Let

$$u = w - \varphi \tag{2.7}$$

in (2.6), where φ is an arbitrary harmonic function in $\mathscr{D}$*. Then

$$\begin{aligned} |w(P) - \varphi(P)|^2 \leqq{} & K_0 M^{2(1-\delta)} \Big\{ K_1 \int_{\Sigma} [(w - f) - (\varphi - f)]^2 \, d\sigma + \\ & + K_2 \int_{\Sigma} [(w_{,i} - F_i) - (\varphi_{,i} - F_i)] \, [(w_{,i} - F_i) - (\varphi_{,i} - F_i)] \, d\sigma \Big\}^{\delta} \\ \leqq{} & 2^{\delta} K_0 M^{2(1-\delta)} \Big\{ K_1 \Big[\alpha_1 + \int_{\Sigma} (\varphi - f)^2 \, d\sigma \Big] + \\ & + K_2 \Big[\alpha_2 + \int_{\Sigma} (\varphi_{,i} - F_i)(\varphi_{,i} - F_i) \, d\sigma \Big] \Big\}^{\delta}. \end{aligned} \tag{2.8}$$

Thus if α_1 and α_2 are small and φ can be chosen to approximate the data of w, then it is possible to obtain close upper and lower bounds for $w(P)$. The usual procedure for approximating the data is to choose

$$\varphi = \sum_{i=1}^{N} b_i \varphi_i \tag{2.9}$$

(where the φ_i are N independent harmonic functions) and use the Rayleigh-Ritz technique to minimize the right-hand side of (2.8). Thus, once we have obtained (2.6), pointwise estimates for $w(P)$ follow immediately. We turn now to the derivation of (2.6).

3. Preliminary Estimates

For $n > 2$ we define for any function u a functional in the following way:

$$V(r, u) = \log F(r, u) + C r^{-2(n-2)} \tag{3.1}$$

where

$$F(r, u) = \int_{r_0}^{r} \varrho^{1-n} A_{\varrho}(u) \, d\varrho + K_1 \varepsilon_1 + K_2 \varepsilon_2 \tag{3.2}$$

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* It is clear that if w is uniformly bounded by some constant M_1 in $\mathscr{D}$ and φ is a prescribed function, the constant M in (2.5), with u defined by (2.7), can be computed easily.

and

$$C = \frac{(\sqrt{2}+1)\, r_0^{n-1} R^{n-3}}{2(n-2)^2},$$

$$K_1 = \frac{n}{2(n-2)\, r_0^{n-1}}, \tag{3.3}$$

$$K_2 = \frac{(n-2)\, R^2 + r_0^2}{2(n-2)\, r_0^{n-1}}.$$

We show that with the choice of constants given by (3.3), $V(r, u)$ is a convex function of $r^{-(n-2)}$. To prove this we form

$$\begin{aligned} &[F(r,u)]^2 \frac{\partial^2 V(r,u)}{(\partial r^{-(n-2)})^2} \\ &\quad = (r-2)^{-2} \Big\{ r^{n-1} F(r,u) \int\limits_{\omega(r)} u_{,i} u_{,i} \, ds - [A_r(u)]^2 \Big\} + 2C\, [F(r,u)]^2. \end{aligned} \tag{3.4}$$

Let Σ' denote the portion of Σ lying inside $K(r)$; then since u is harmonic

$$\begin{aligned} F(r,u) = K_1 \varepsilon_1 + K_2 \varepsilon_2 + (n-2)^{-1} \int\limits_{\Sigma'} [\varrho^{-(n-2)} - r^{-(n-2)}]\, u \frac{\partial u}{\partial n} \, d\sigma + \\ + \frac{1}{2} r^{-(n-1)} \int\limits_{\omega(r)} u^2 \, ds + \frac{1}{2} \int\limits_{\Sigma'} \varrho^{-(n-1)} \frac{\partial \varrho}{\partial n} u^2 \, d\sigma \end{aligned} \tag{3.5}$$

where $\partial/\partial n$ denotes the outward normal derivative on Σ', and ϱ measures the distance from the origin P_0. Clearly

$$\begin{aligned} &\left[K_1 - \frac{r_0^{-(n-1)}}{(n-2)}\right] \varepsilon_1 + \left[K_2 - \frac{r_0^{-(n-3)}}{2(n-2)}\right] \varepsilon_2 \\ &\quad \leqq F(r,u) - \frac{r^{-(n-1)}}{2} \int\limits_{\omega(r)} u^2 \, ds \leqq \left[K_1 + \frac{r_0^{-(n-1)}}{(n-2)}\right] \varepsilon_1 + \left[K_2 + \frac{r_0^{-(n-3)}}{2(n-2)}\right] \varepsilon_2. \end{aligned} \tag{3.6}$$

Also by GREEN'S identity and the Schwarz inequality we have

$$\begin{aligned} [A_r(u)]^2 &= \Big\{ \int\limits_{\omega(r)} u \frac{\partial u}{\partial r} \, ds + \int\limits_{\Sigma'} u \frac{\partial u}{\partial n} \, d\sigma \Big\}^2 \\ &\leqq \int\limits_{\omega(r)} u^2 \, ds \int\limits_{\omega(r)} \left(\frac{\partial u}{\partial r}\right)^2 ds + \\ &\quad + \frac{1}{2} \Big[R^{-1} \int\limits_{\omega(r)} u^2 \, ds + R \int\limits_{\omega(r)} u_{,i} u_{,i} \, ds \Big] [R^{-1} \varepsilon_1 + R\, \varepsilon_2] + \varepsilon_1 \varepsilon_2. \end{aligned} \tag{3.7}$$

We need also the inequality

$$\int\limits_{\omega(r)} \left[|\operatorname{grad}_s u|^2 - \left(\frac{\partial u}{\partial r}\right)^2 \right] ds \geqq -\sqrt{2}\, \varepsilon_2 \tag{3.8}$$

where grad_s denotes the tangential projection of the gradient. Inequality (3.8) follows directly from the RELLICH identity [*20*], which for the region $\mathscr{D}_r$ reduces to

$$\begin{aligned} r \int\limits_{\omega(r)} \left[|\operatorname{grad}_s u|^2 - \left(\frac{\partial u}{\partial r}\right)^2 \right] ds = 2 \int\limits_{\Sigma'} \varrho \operatorname{grad}_s \varrho \cdot \operatorname{grad}_s u \frac{\partial u}{\partial n} \, d\sigma - \\ - \int\limits_{\Sigma'} \varrho \frac{\partial \varrho}{\partial n} \left[|\operatorname{grad}_s u|^2 - \left(\frac{\partial u}{\partial n}\right)^2 \right] d\sigma + (n-2)\, A_r(u). \end{aligned} \tag{3.9}$$

This identity has been used previously for obtaining bounds in Dirichlet and Neumann problems [*15*]. We now insert (3.6), (3.7) and (3.8) into (3.4), *i.e.*,

$$\begin{aligned}
[F(r,u)]^2 \frac{\partial^2 V}{(\partial r^{-(n-2)})^2} \geqq (n-2)^{-2} \Bigg\{ & \frac{1}{2} \int\limits_{\omega(r)} u^2\,ds \int\limits_{\omega(r)} \left[|\operatorname{grad}_s u|^2 - \left(\frac{\partial u}{\partial r}\right)^2 \right] ds + \\
& + \left(\left[K_1 - \frac{r_0^{-(n-1)}}{(n-2)} \right] \varepsilon_1 + \left[K_2 - \frac{r_0^{-(n-3)}}{2(n-2)} \right] \varepsilon_2 \right) r^{n-1} \int\limits_{\omega(r)} u_{,i}\,u_{,i}\,ds - \\
& - \frac{1}{2} \left[R^{-1} \int\limits_{\omega(r)} u^2\,ds + R \int\limits_{\omega(r)} u_{,i}\,u_{,i}\,ds \right] [R^{-1}\varepsilon_1 + R\,\varepsilon_2] - \varepsilon_1\varepsilon_2 \Bigg\} + \\
& + 2C \left[\frac{r^{-(n-1)}}{2} \int\limits_{\omega(r)} u^2\,ds + \left(K_1 - \frac{r_0^{-(n-1)}}{(n-2)} \right) \varepsilon_1 + \left(K_2 - \frac{r_0^{-(n-3)}}{2(n-2)} \right) \varepsilon_2 \right]^2 .
\end{aligned} \tag{3.10}$$

If we replace the second integral on the right by $-\sqrt{2}\,\varepsilon_2$ and make use of (3.3), we find that the right-hand side is non-negative. It follows then that $V(r,u)$ is a convex function of $r^{-(n-2)}$.

It is clear that the values for C, K_1, and K_2 given in (3.3) will not in general be the best possible values. The best choice will depend on the geometry of Σ, the values of ε_1 and ε_2, the number of dimensions and the location of P_0. For instance, instead of the left-hand portion of the inequality (3.6) one could have written

$$F(r,u) - \frac{r^{-(n-1)}}{2} \int\limits_{\omega(r)} u^2\,ds \geqq \left[K_1 - \frac{(\alpha+1)\,r_0^{-(n-1)}}{2(n-2)} \right] \varepsilon_1 + \left[K_2 - \frac{r_0^{-(n-3)}}{2(n-2)\,\alpha} \right] \varepsilon_2 \tag{3.11}$$

where α is an arbitrary positive constant. Similarly (3.7) could have been replaced by

$$\begin{aligned}
[A_r(u)]^2 \leqq & \int\limits_{\omega(r)} u^2\,ds \int\limits_{\omega(r)} \left(\frac{\partial u}{\partial r}\right)^2 ds + \\
& + \tfrac{1}{2} \left[(\beta R)^{-1} \int\limits_{\omega(r)} u^2\,ds + \beta R \int\limits_{\omega(r)} u_{,i}\,u_{,i}\,ds \right] [(\gamma R^{-1})\,\varepsilon_1 + \gamma R\,\varepsilon_2] + \varepsilon_1\varepsilon_2
\end{aligned} \tag{3.12}$$

where β and γ are arbitrary positive constants. Thus, C, K_1 and K_2 could have been made to depend on the three arbitrary parameters α, β, and γ.

In two dimensions we define $V(r,u)$ as

$$V(r,u) = \log \left| \int\limits_{r_0}^{r} \varrho^{-1} A_\varrho(u)\,d\varrho + K_1'\,\varepsilon_1 + K_2'\,\varepsilon_2 \right| + C'(\log r)^2 \tag{3.13}$$

where C', K_1', and K_2' are given by

$$\begin{aligned}
C' &= \tfrac{1}{2}(1+\sqrt{2})\,r_0 R^{-1}, \\
K_1' &= \tfrac{1}{2}(3 r_0)^{-1}, \\
K_2' &= \tfrac{1}{2}[r_0^2 \log^2 (R/r_0) + R^2]\, r_0^{-1}.
\end{aligned} \tag{3.14}$$

It is easily verified that $V(r,u)$ given by (3.13) is then a convex function of $\log r$.

We assume again that $n>2$ (the same procedure applied to (3.13) will yield similar results for $n=2$). From the convexity of $V(r, u)$ for $r_0 \leqq r \leqq R$ it follows that

$$\begin{aligned} &\exp[C\, r^{-2(n-2)}]\, F(r, u) \\ &\qquad \leqq \{\exp[C\, R^{-2(n-2)}]\, F(R, u)\}^{1-\delta} \{\exp[C\, r_0^{-2(n-2)}\, F(r_0, u)\}^{\delta} \end{aligned} \tag{3.15}$$

where

$$\delta = \left(\frac{r_0}{r}\right)^{n-2} (R^{n-2} - r^{n-2})/(R^{n-2} - r_0^{n-2}). \tag{3.16}$$

But from (3.1), (3.3) and (3.6) we see that

$$F(R, u) \leqq \left\{\frac{M^2\, S(R)}{2R^{n-1}} + \frac{(n+2)\, r^{-(n-1)}}{2(n-2)}\, \varepsilon_1 + \frac{[(n-2)\, R^2 + 2r_0^2]}{2(n-2)\, r_0^{n-1}}\, \varepsilon_2\right\} = M_1^2 \tag{3.17}$$

where $S(R)$ is the area of $\omega(R)$. Note that when upper bounds for ε_1 and ε_2 are known then M_1 is completely determined. Since these upper bounds will be computed*, let us write

$$\varepsilon_1 \leqq C_1\, M^2, \qquad \varepsilon_2 \leqq C_2\, M^2 \tag{3.18}$$

where C_1 and C_2 are known constants. Then instead of M_1^2 we write

$$M_1^2 \leqq B_1\, M^2 \tag{3.19}$$

where

$$B_1 = \frac{S(R)}{2R^{n-1}} + \frac{(n+2)\, C_1}{2(n-2)\, r_0^{n-1}} + \frac{[(n-2)^2\, R^2 + 2r_0^2]}{2(n-2)\, r_0^{n-1}}\, C_2. \tag{3.20}$$

From (3.2) it is obvious that

$$F(r_0, u) = K_1\, \varepsilon_1 + K_2\, \varepsilon_2. \tag{3.21}$$

Thus (3.15) may be rewritten as

$$F(r, u) \leqq B\, M^{2(1-\delta)}\, (K_1\, \varepsilon_1 + K_2\, \varepsilon_2)^{\delta} \tag{3.22}$$

where

$$B = B_1^{1-\delta} \exp\{C\, [(1-\delta)\, R^{-2(n-2)} + \delta\, r_0^{-2(n-2)} - r^{-2(n-2)}\}. \tag{3.23}$$

Using (3.6), we then have

$$r^{-(n-1)} \int_{\omega(r)} u^2\, ds + r_0^{-(n-1)}\, \varepsilon_1 + R^2\, r_0^{-(n-1)}\, \varepsilon_2 \leqq 2B\, M^{2(1-\delta)}\, (K_1\, \varepsilon_1 + K_2\, \varepsilon_2)^{\delta}. \tag{3.24}$$

Note that a somewhat simpler expression arises if we know instead of (2.5) a bound for $A(u)$, *i.e.*,

$$A(u) = \int_{\mathcal{D}} u_{,i}\, u_{,i}\, dx \leqq L^2. \tag{3.25}$$

This condition is obviously sufficient to insure stability. We have then instead of (3.17)

$$F(R, u) \leqq \left[\frac{L^2}{(n-2)}\, (r_0^{-(n-2)} - R^{-(n-2)}) + K_1\, \varepsilon_1 + K_2\, \varepsilon_2\right]. \tag{3.26}$$

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* See the last paragraph of Section 4.

Thus, one could write instead of (3.24) the expression

$$(3.27)\qquad \begin{aligned} r^{-(n-1)} \int_{\omega(r)} u^2\, ds + r_0^{-(n-1)}\, \varepsilon_1 + R^2\, r_0^{-(n-1)}\, \varepsilon_2 \\ \leqq C_1'\, L^{2(1-\delta)} (K_1\, \varepsilon_1 + K_2\, \varepsilon_2)^\delta + C_2' (K_1\, \varepsilon_1 + K_2\, \varepsilon_2) \end{aligned}$$

where C_1' and C_2' are easily determined.

It is clear also that stability and pointwise bounds will follow if we use a bound of the following type:

$$(3.28)\qquad \int_{\mathscr{D}} |u|\, |\operatorname{grad} u|\, dv \leqq L_1^2$$

instead of (2.5) or (3.25).

4. Pointwise Estimates

We derive first an *a priori* estimate for the value of u at a point P in $\mathscr{D}_r$. Let $G(P, Q)$ be the harmonic GREEN'S function for the sphere $K(r)$ with singularity at P. Then by GREEN'S formula we have

$$(4.1)\qquad u(P) = -\int_{\omega(r)} u \frac{\partial G}{\partial n}\, ds + \int_{\Sigma'} G \frac{\partial u}{\partial n}\, d\sigma - \int_{\Sigma'} u \frac{\partial G}{\partial n}\, d\sigma.$$

If the point P is not a boundary point of $\mathscr{D}_r$, equation (4.1) may be differentiated any number of times with respect to the variables of the point P, the order of differentiation and integration being interchangeable on the right-hand side. Using SCHWARZ'S inequality we obtain

$$(4.2)\qquad \begin{aligned} |u(P)|^2 \leqq \left[r^{-(n-1)} \int_{\omega(r)} u^2\, ds + r_0^{-(n-1)}\, \varepsilon_1 + R^2\, r_0^{-(n-1)}\, \varepsilon_2 \right] \times \\ \times \left[r^{n-1} \int_{\omega(r)} \left(\frac{\partial G}{\partial n} \right)^2 ds + r_0^{n-1} \int_{\Sigma'} \left(\frac{\partial G}{\partial n} \right)^2 d\sigma + r_0^{n-1} R^{-2} \int_{\Sigma'} G^2\, d\sigma \right]. \end{aligned}$$

If we replace the first bracket term on the right using (3.24), we find

$$(4.3)\qquad \begin{aligned} |u(P)|^2 \leqq 2B\, M^{2(1-\delta)} (K_1\, \varepsilon_1 + K_2\, \varepsilon_2)^\delta \times \\ \times \left[r^{n-1} \int_{\omega(r)} \left(\frac{\partial G}{\partial n} \right)^2 ds + r_0^{n-1} \int_{\Sigma'} \left(\frac{\partial G}{\partial n} \right)^2 d\sigma + r_0^{n-1} R^{-2} \int_{\Sigma'} G^2\, d\sigma \right]. \end{aligned}$$

Inequality (4.3) thus gives an *a priori* estimate for the value of u at a point P in $\mathscr{D}_r$ which is of the desired type (2.6). It depends on the uniform bound M and on the Cauchy data in such a way that if the data tends to zero the value of $u(P)$ tends to zero.

We denote by ξ_i the variables of the point P. Then by differentiation of (4.1) we obtain

$$(4.4)\qquad u_{,i}(P) = -\int_{\omega(r)} u \frac{\partial}{\partial n} G_{,i}\, ds + \int_{\Sigma'} G_{,i} \frac{\partial u}{\partial n}\, d\sigma - \int_{\Sigma'} u \frac{\partial}{\partial n} G_{,i}\, d\sigma$$

where the comma in this case denotes differentiation with respect to the variable ξ_i. If P is not a boundary point of $\mathscr{D}_r$, we again apply the Schwarz inequality and obtain a bound for any derivative of u at the point P. Since we shall need it later, we write down the estimate for the gradient of u at P.

$$\begin{aligned}[u_{,i}(P)\, u_{,i}(P)] &\leqq 2B\, M^{2(1-\delta)} (K_1\, \varepsilon_1 + K_2\, \varepsilon_2)^\delta \times \\ &\times \left\{ r^{n-1} \int\limits_{\omega(r)} \frac{\partial}{\partial n} G_{,i} \frac{\partial}{\partial n} G_{,i}\, ds + r_0^{n-1} \int\limits_{\Sigma'} \left[\frac{\partial}{\partial n} G_{,i} \frac{\partial}{\partial n} G_{,i} + R^{-2} G_{,i} G_{,i} \right] d\sigma \right\}.\end{aligned} \tag{4.5}$$

Suppose now that we wish to obtain bounds for the value of u at a point P in $\mathscr{D} - \mathscr{D}_r$. We proceed in the following way: Let $\mathscr{D}^*$ be a subdomain of $\mathscr{D}$ whose boundary $\mathscr{C}^*$ coincides with $\mathscr{C}$ except for a small band $r_1 + \gamma \geqq \varrho \geqq r_1 - \gamma$ where γ is a small positive constant and $r > r_1 > r_0$. If $\omega^*(r_1)$ is defined relative to $\mathscr{C}^*$ as $\omega(r_1)$ was relative to $\mathscr{C}$, we prescribe that $\mathscr{D}^*$ be so constructed that $\omega^*(r_1)$ and $\mathscr{C}$ have no points in common. In practice the sphere $K(r_1)$ will first be decided upon, and then $\mathscr{C}^*$ will be determined. We now integrate (4.3) and (4.5) over the surface $\omega^*(r_1)$, *i.e.*,

$$\begin{aligned}\eta_1 = \int\limits_{\omega^*(r_1)} u^2\, ds &\leqq 2B\, M^{2(1-\delta)} (K_1\, \varepsilon_1 + K_2\, \varepsilon_2) \times \\ &\times \left\{ r^{n-1} \int\limits_{\omega^*(r_1)} \int\limits_{\omega(r)} \left(\frac{\partial G}{\partial n}\right)^2 ds\, ds^* + r_0^{n-1} \int\limits_{\omega^*(r_1)} \int\limits_{\Sigma'} \left[\left(\frac{\partial G}{\partial n}\right)^2 + R^{-2} G^2 \right] d\sigma\, ds^* \right\}\end{aligned} \tag{4.6}$$

and

$$\begin{aligned}\eta_2 = \int\limits_{\omega^*(r_1)} u_{,i}\, u_{,i}\, ds &\leqq 2B\, M^{2(1-\delta)} (K_1\, \varepsilon_1 + K_2\, \varepsilon_2)^\delta \times \\ &\times \left\{ r^{n-1} \int\limits_{\omega^*(r_1)} \int\limits_{\omega(r)} \frac{\partial G_{,i}}{\partial n} \frac{\partial G_{,i}}{\partial n}\, ds\, ds^* + \right. \\ &\left. + r_0^{n-1} \int\limits_{\omega^*(r_1)} \int\limits_{\Sigma'} \left[\frac{\partial G_{,i}}{\partial n} \frac{\partial G_{,i}}{\partial n} + R^{-2} G_{,i} G_{,i} \right] d\sigma\, ds^* \right\}.\end{aligned} \tag{4.7}$$

Note that instead of (4.6) we could use (3.24) for $r = r_1$ and the fact that

$$\int\limits_{\omega^*(r_1)} u^2\, ds^* < \int\limits_{\omega(r)} u^2\, ds. \tag{4.8}$$

The object of introducing the subdomain $\mathscr{D}^*$ was merely to avoid integrating all the way to the boundary of $\mathscr{D}_r$ in (4.6) and (4.7). For then the integrals on the right would not have converged. Actually we do not use $\mathscr{D}^*$ or $\mathscr{C}^*$ but only $\omega^*(r_1)$. Consequently another way of stating what we have done is to say that we formed (4.6) and (4.7) by integrating (4.3) and (4.5) over a portion $\omega^*(r_1)$ of the surface of the sphere $K(r_1)$ (where $r > r_1 > r_0$) with the stipulation that every point of $\omega^*(r_1)$ should be in $\mathscr{D}_r$ at a distance greater than some number β from $\mathscr{C}$.

It is obvious from (4.6) and (4.7) that we may regard the surface $\omega^*(r_1)$ as a Cauchy surface, that is, one on which inequalities for the Cauchy data are known. Our object is to obtain an estimate for the value of u at a point P in $\mathscr{D}^* - \mathscr{D}_{r_1}^*$ with boundary $\mathscr{C}_1^*$. Thus we consider the following problem:

$$\Delta u = 0 \quad \text{in} \quad \mathscr{D}^* - \mathscr{D}_{r_1}^* \tag{4.9}$$

with upper bounds for η_1 and η_2 known on $\omega^*(r_1)$. We proceed as before, letting $\omega^*(r_1)$ be our new surface Σ^* and letting $\mathscr{C}_1^* - \Sigma^*$ be the new $\bar{\Sigma}^*$. We choose a new origin P_0^* to have the same properties with respect to the new $\bar{\Sigma}^*$ that P_0 possessed for the original $\bar{\Sigma}$. We define new quantities R^* and r_0^* in the obvious way, and, proceeding as in Sections 3 and 4, we are able to obtain bounds for the value of u and its derivatives at any point P^* in $\mathcal{D}^*_{r^*}$. (By $\mathcal{D}^*_{r^*}$ we mean the intersection of $\mathcal{D}^* - \mathcal{D}^*_{r_1}$ and $K(r^*)$ with center at P_0^*.) These bounds will be given by (4.3) and (4.5) with the coordinates changed to starred coordinates, the constants B, K_1 and K_2 replaced by constants B^*, K_1^* and K_2^*, and ε_1 and ε_2 replaced by η_1 and η_2. For example

$$|u(P^*)|^2 \leqq 2B^* M^{2(1-\delta^*)} (K_1^* \eta_1 + K_2^* \eta_2)^{\delta^*} \times \tag{4.10}$$
$$\times \left\{ (r^*)^{n-1} \int\limits_{\omega^*(r^*)} \left(\frac{\partial G^*}{\partial n}\right)^2 ds^* + (r_0^*)^{n-1} \int\limits_{\Sigma^{*\prime}} \left[\left(\frac{\partial G^*}{\partial n}\right)^2 + (R^*)^{-2} (G^*)^2\right] d\sigma^* \right\}$$

with η_1 and η_2 defined by (4.6) and (4.7). It is obvious that (4.10) is of the form

$$|u(P^*)|^2 \leqq K^* M^{2(1-\delta\delta^*)} (K_1 \varepsilon_1 + K_2 \varepsilon_2)^{\delta\delta^*} \tag{4.11}$$

where the K^* is a determined quantity.

If we wish to obtain bounds for the value of u at a point which is neither in $\mathcal{D}_r$ nor in $\mathcal{D}^*_{r^*}$, we again define a new Cauchy surface, and in fact continue to define new such Cauchy surfaces until we arrive at the point at which bounds are desired. It is obvious that there is considerable arbitrariness in this procedure, and the optimum estimates will depend on the geometry of Σ, the values of ε_1 and ε_2, *etc.*

One can also obtain in an obvious way pointwise estimates for any derivatives of u in $\mathcal{D}$. It is likewise possible to obtain pointwise estimates for u and its derivatives if the region is unbounded. No conditions need be imposed on the behavior of u at infinity.

We note that in obtaining pointwise estimates for the solution of (2.1) and (2.2) we subsequently set $u = w - \varphi$, choose the φ as indicated in (2.9) and minimize with respect to the coefficients. Regardless of the number of intermediate Cauchy surfaces it may have been necessary to define, the constants b_i in (2.9) must always be chosen to minimize the expression

$$K_1\left[\alpha_1 + \int\limits_{\Sigma} (\varphi - f)^2 d\sigma\right] + K_2\left[\alpha_2 + \int\limits_{\Sigma} (\varphi_{,i} - F_i)(\varphi_{,i} - F_i)\, d\sigma\right] \tag{4.12}$$

occurring in (2.8). Thus one starts by choosing a large enough number N of harmonic functions φ_i to make the expression (4.12) sufficiently small. The constants C_1 and C_2 of (3.18) are then easily computed. The determination of the constant K_0 in (2.6) is then straightforward.

5. Practical Application of Results

It is apparent from the procedure described in the preceding section that the precision with which we can approximate the solution of (2.1) and (2.2) at a point P with a given set of functions φ_i depends very much on the number of Cauchy surfaces which we must define in order to arrive at P. Thus if Σ is sufficiently small or if the domain $\mathcal{D}$ is sufficiently thin or irregular (for example, a region in the shape of a dumbbell with Cauchy data given on a portion of one of the knobs), it may not be possible to get practical bounds for u at a point sufficiently far removed from Σ. In many physical problems, however, the point P may be used to determine Σ. If we wish to determine certain steady state atmospheric conditions at a point P in the atmosphere, we take measurements of Cauchy data over a portion Σ of the surface of the earth which is so large that the point P will lie inside the original $\mathscr{D}_r$. A number of interesting problems in electrostatics and magnetostatics, as well as gravitational potential problems, may be handled in a similar way. It should perhaps be pointed out that the gravitational potential v satisfies a Poisson equation, but if we approximate with functions ψ satisfying the same equation, then the quantity $u = v - \psi$ will be a harmonic function.

The results of this paper may likewise be used to investigate the field (flow, electrostatic, magnetostatic, *etc.*) which gives rise to a given free surface.

Also in certain boundary value problems in which the point P lies close to the boundary it may often be more convenient to measure Cauchy data quite accurately over a portion of the boundary near P and use the results of this paper, rather than to consider the standard boundary value problem. This is particularly true if the region is extremely large or if portions of the boundary are irregular.

It may happen that isolated singularities (or singular surfaces of dimension less than $n-2$ in general) occur in $\mathscr{D}$. In theory one can obtain bounds for $u(P)$ at any point P outside a small neighborhood of the point at which the singularity occurs. In practise one may not be able to obtain close bounds at each such point P unless the constants α_1 and α_2 in (2.2) are extremely small and the number N in (2.9) is extremely large.

In other problems the portion of the boundary $\overline{\Sigma}$ may not be known. One may wish to choose it to be a level surface for a larger (possibly infinite) domain as is the case in certain design problems. The method developed here may be applied to such a problem.

If it were possible to measure the Cauchy data exactly, and if Σ were sufficiently smooth and we knew uniform bounds for a sufficient number of derivatives of u, then the procedure could be simplified considerably. We could actually find a bound similar to (3.24) for $\int\limits_{\omega(r)} |\operatorname{grad} u|^2\, ds$, and it would not be necessary to define $\mathscr{D}^*$. However, it is extremely unlikely that this will ever be the case in a real physical problem. Also if we could obtain bounds of the type (2.2) for higher derivatives of u on Σ, and if Σ were sufficiently smooth, then we could again find a bound similar to (3.24) for $\int\limits_{\omega(r)} |\operatorname{grad} u|^2\, ds$. It would again be unnecessary to define $\mathscr{D}^*$. However, one must expect poor accuracy in the measurement of higher derivatives of u on Σ, so that the net result would usually be a worse bound for $u(P)$ than would be obtained by the method suggested in this paper.

This research was supported in part by the United States Air Force through the Air Force Office of Scientific Research of the Air Research and Development Command under Contract No. AF 18(600)-573 and in part by the National Science Foundation through a postdoctoral fellowship.

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(Received January 15, 1960)

Über die Randkomponenten ebener harmonischer Vektorfelder

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§ 1. Einleitung

Bei verschiedenen physikalischen Anwendungen der Potentialtheorie spielen harmonische Vektorfelder[1] eine Rolle, die auf und außerhalb einer bestimmten Berandung in der (x, y)-Ebene erklärt und im Unendlichen beschränkt sind. Die übliche Methode, solche Vektorfelder aus gewissen Randbedingungen zu berechnen, besteht in der Darstellung des Feldes als (negativer) Gradient eines skalaren oder als Rotation eines (bei ebenen Problemen nur eine Komponente besitzenden) Vektorpotentials.

Über die Einwertigkeit solcher Potentiale ist zunächst noch nichts ausgesagt. Man weiß jedoch, daß ein harmonisches Feld im Falle verschwindender Zirkulation (Linienintegral der Tangentialkomponente) ein einwertiges skalares und im Falle verschwindender Ergiebigkeit (Linienintegral der Normalkomponente) ein einwertiges Vektorpotential besitzt. Bekannte Beispiele hierfür sind das (zirkulationsfreie) elektrostatische Feld eines Zylinders mit einer von z unabhängigen Ladungsdichte und das (ergiebigkeitsfreie) magnetostatische Feld eines stromdurchflossenen Zylinders mit von z unabhängiger Stromdichte [*1*]. Die mathematische Physik kennt jedoch auch Beispiele harmonischer Vektorfelder, die im allgemeinen weder ergiebigkeits- noch zirkulationsfrei sind, bei denen also beide Potentiale im allgemeinen notwendig mehrwertig sind. Dazu zählt das Ausblasungs- und Absaugungsproblem der Aerodynamik [*2*]. Man sucht darin die inkompressible Potentialströmung zu bestimmen, die sich auf und außerhalb eines im Parallelstrom befindlichen Tragflügelprofils unendlicher Streckung ausbildet, wenn auf dem Profil eine „Ergiebigkeitsverteilung", d.h. die Normalkomponente vorgeschrieben ist. Hier werden beide Potentiale[2] in dem Falle mehrwertig, wenn außer der Ergiebigkeit auch die den Auftrieb erzeugende Zirkulation von Null verschieden ist.

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[1] Unter einem harmonischen Vektorfeld wird ein divergenz- und rotationsfreies Feld verstanden. Es folgt hieraus (in kartesischen Koordinaten) zwar, daß jede Komponente eines solchen Feldes auch eine harmonische Funktion ist, d.h. der Differentialgleichung von LAPLACE genügt, jedoch ist nicht jeder Vektor ein harmonischer, dessen Komponenten harmonische Funktionen sind.

[2] Die einzige Komponente des Vektorpotentials wird in der Strömungslehre als Stromfunktion bezeichnet.

Von besonderer Bedeutung ist die Größe des betrachteten Vektorfeldes auf der Berandung selbst, z.B. die elektrische Feldstärke auf der Oberfläche eines Leiters oder der Vektor der Strömungsgeschwindigkeit auf der Oberfläche eines Tragflügels, und zwar teils, weil von der physikalischen Problemstellung her oftmals nur die Randkomponenten des Feldes interessant sind und teils, weil das Feld im Außengebiet auf Grund bekannter Sätze der Potentialtheorie durch die Randwerte bestimmt ist. Die Untersuchung harmonischer Vektorfelder läßt sich daher zurückführen auf das äquivalente Problem, die „Verträglichkeiten" der Randkomponenten (Normal- und Tangentialkomponente) zu bestimmen, die, wie da man weiß, nicht voneinander unabhängig sind.

In der vorliegenden Arbeit werden ebene harmonische Vektorfelder im Außengebiet beliebiger Berandungen, die im wesentlichen nur gewissen Stetigkeits- und Differenzierbarkeitsvoraussetzungen genügen müssen, betrachtet. Ergiebigkeit und Zirkulation dürfen grundsätzlich von Null verschieden sein. Es wird der Zusammenhang zwischen der Normal- und der Tangentialkomponente auf der Berandung untersucht, und im Falle der Vorgabe einer Komponente werden entsprechende Existenz- und Eindeutigkeitssätze formuliert. Im Hinblick auf praktische Anwendungen wird ein numerisches Berechnungsverfahren angegeben. Schließlich wird gezeigt, daß sich diese Überlegungen ohne Schwierigkeiten auf die in der Hydrodynamik vorkommenden „Gitterprofile" übertragen lassen.

Die erhaltenen Ergebnisse kann man auf folgende einfache Weise von der Anschauung her verstehen. Sei eine beliebige Ergiebigkeitsverteilung auf einem Kreis vorgeschrieben, so läßt sich die Gesamtheit aller im Außengebiet einschließlich des Unendlichen möglichen, inkompressiblen Potentialströmungen bekanntlich aus vier „Grundströmungen" linear superponieren; diese sind im einzelnen eine im Unendlichen verschwindende „reine Zirkulationsströmung", die den Kreis als Stromlinie enthält, ferner je eine zirkulationsfreie, den Kreis als Verzweigungsstromlinie enthaltende Strömung, die im Unendlichen bzw. nur eine x- oder y-Komponente besitzt, und schließlich eine im Unendlichen verschwindende, zirkulationsfreie Strömung, die den Kreis in der Weise durchsetzt, daß die vorgegebene Normalkomponente angenommen wird. Auf dem Kreis selbst sind die zugehörigen Tangentialkomponenten dieser vier Anteile hinlänglich bekannt bzw. mit Hilfe des Poissonschen Integrals geschlossen darstellbar [*3*]. Es wird nun gezeigt, daß die für den Kreis bestehenden Zusammenhänge auch für beliebige Berandungen unverändert gültig bleiben. Allerdings sind die praktisch interessierenden Tangentialkomponenten der genannten vier Grundströmungen nicht mehr in geschlossener Form angebbar, wohl aber als Lösungen von Integralgleichungen zweiter Art mit stetigem Kern ohne Schwierigkeiten numerisch zu gewinnen.

§ 2. Die betrachtete Funktionenklasse

In der (x, y)-Ebene sei eine stetig gekrümmte, doppelpunktfreie, geschlossene Kurve $\mathfrak{C}$ gegeben. Es ist immer ohne Einschränkung möglich, für $\mathfrak{C}$ einen Kurvenparameter φ derart anzugeben, daß $x(\varphi)$, $y(\varphi)$ für alle φ zweimal stetig differenzierbar sind, die Periode 2π besitzen und $\mathfrak{C}$ mit wachsendem $0 \leqq \varphi \leqq 2\pi$ genau einmal im Gegenuhrzeigersinn durchlaufen und daß außerdem die ersten Ableitungen $\dot{x}(\varphi)$, $\dot{y}(\varphi)$ nirgends gleichzeitig verschwinden.

Gegenstand der Betrachtung sind alle auf und außerhalb $\mathfrak{C}$ erklärten Vektorfunktionen $\mathfrak{v}(x, y) = (v_x(x, y), v_y(x, y))$ mit folgenden Eigenschaften:

$\mathfrak{v}$ ist auf und außerhalb $\mathfrak{C}$ stetig;

$\mathfrak{v}$ ist außerhalb $\mathfrak{C}$ stetig differenzierbar und harmonisch, d.h. außerhalb $\mathfrak{C}$ gelten die Differentialgleichungen

$$\operatorname{div} \mathfrak{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0, \tag{1}$$

$$\operatorname{rot}_z \mathfrak{v} = \frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} = 0\,[1]; \tag{2}$$

$\mathfrak{v}$ konvergiert im Unendlichen gleichmäßig gegen eine vorgegebene vektorielle Konstante $\mathfrak{v}_\infty = (v_{\infty,x}, v_{\infty,y})$, d.h. für alle $0 \leqq \Theta \leqq 2\pi$ existieren die beiden Grenzwerte

$$\lim_{R \to \infty} \mathfrak{v}(R \cos \Theta, R \sin \Theta) = \mathfrak{v}_\infty \tag{3}$$

im Sinne gleichmäßiger Konvergenz.

Sei v_n die Normalkomponente und v_t die Tangentialkomponente von $\mathfrak{v}$, genommen auf $\mathfrak{C}$[2], so verstehen wir unter der Ergiebigkeit von $\mathfrak{C}$ das Integral

$$E = \int_{\mathfrak{C}} v_n \, ds, \tag{4}$$

während die Zirkulation um $\mathfrak{C}$ gegeben ist durch

$$\Gamma = \int_{\mathfrak{C}} v_t \, ds. \tag{5}$$

Stellt insbesondere $\mathfrak{v}$ eine inkompressible Potentialströmung mit der Dichte ϱ dar, so sind Ergiebigkeit und Zirkulation maßgebend für die auf das Profil $\mathfrak{C}$ ausgeübte Gesamtkraft $\mathfrak{K}$ (Dimension: Kraft/Länge); es gilt nämlich als Folge des Impulssatzes

$$\mathfrak{K} = -\varrho E \mathfrak{v}_\infty + \varrho \Gamma [\mathfrak{v}_\infty, \mathfrak{k}]\,[3]. \tag{6}$$

Demnach bewirkt negative Ergiebigkeit einen Widerstand, d.h. eine Kraftkomponente in Richtung von $\mathfrak{v}_\infty$ und negative Zirkulation einen Auftrieb, d.h. eine Kraftkomponente in der aus $\mathfrak{v}_\infty$ durch Drehung um $+\pi/2$ hervorgehenden Richtung. Für $E = 0$ geht (6) über in den Satz von Kutta-Joukowski.

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[1] Obwohl hier nur ebene Vektorfelder $\mathfrak{v}$ betrachtet werden, ist es gelegentlich von Vorteil, $\mathfrak{v}$ durch eine identisch verschwindende z-Komponente formal zu einem dreidimensionalen Vektor zu ergänzen; insbesondere besitzt dann der (dreidimensionale) Vektor $\operatorname{rot} \mathfrak{v}$ die identisch verschwindenden Komponenten $\operatorname{rot}_x \mathfrak{v}$ und $\operatorname{rot}_y \mathfrak{v}$.

[2] Wenn nicht ausdrücklich etwas Gegenteiliges gesagt wird, soll $\mathfrak{n}$ der Einheitsvektor in Richtung der äußeren Normalen und $\mathfrak{t}$ der im Gegenuhrzeigersinn orientierte Tangentialeinheitsvektor einer geschlossenen Kurve sein; wir verstehen unter der Normal- und Tangentialkomponente v_n und v_t bzw. die Skalarprodukte $(\mathfrak{n}, \mathfrak{v})$ und $(\mathfrak{t}, \mathfrak{v})$ und unter $\partial/\partial n$ und $\partial/\partial t$ bzw. die Differentiationen in Richtung von $\mathfrak{n}$ und $\mathfrak{t}$. Die Einheitstangente $\mathfrak{t}$ geht aus der Einheitsnormalen $\mathfrak{n}$ durch Drehung um $+\pi/2$ hervor.

[3] $\mathfrak{i}$, $\mathfrak{j}$, $\mathfrak{k}$ sind die Einheitsvektoren in x-, y- und z-Richtung.

§ 3. Darstellung harmonischer Vektorfelder durch ihre Randwerte

Lemma. *Es sei* $\mathfrak{G}$ *ein* p*-fach zusammenhängendes Gebiet der* (x, y)*-Ebene mit stückweise glatten Berandungen* $\mathfrak{S}_i$, $i=1 \ldots p$. *Einheitsnormale* $\mathfrak{n}$ *und Einheitstangente* $\mathfrak{t}$ *der Berandungen seien so orientiert, daß* $\mathfrak{n}$ *aus* $\mathfrak{G}$ *herausgerichtet ist und* $\mathfrak{G}$ *bei Umlaufung in Richtung von* $\mathfrak{t}$ *zur Linken liegt. Für alle Aufpunkte* (x, y) *aus* $\mathfrak{G}$ *und alle Integrationspunkte* $(\xi, \eta) \neq (x, y)$ *aus* $\mathfrak{G}$ *und* $\mathfrak{S}_i$, $i=1 \ldots p$, *sei* $F(x, y; \xi, \eta) = f(x-\xi, y-\eta)$ *als stetige Funktion erklärt; für alle* $(\xi, \eta) \neq (x, y)$ *aus* $\mathfrak{G}$ *sei* F *ferner zweimal stetig differenzierbar und harmonisch. Ferner soll* $F' = F - \ln \frac{1}{r}$ *für alle* (ξ, η) *aus einer Umgebung von* (x, y) *stetig differenzierbar sein, wenn* r *den Abstand zwischen Aufpunkt und Integrationspunkt bedeutet. Dann läßt sich ein in* $\mathfrak{G}$ *und auf* $\mathfrak{S}_i$, $i=1 \ldots p$, *stetiges und in* $\mathfrak{G}$ *stetig differenzierbares und harmonisches Vektorfeld* $\mathfrak{v} = (v_x, v_y)$ *in* $\mathfrak{G}$ *gemäß*

$$\mathfrak{v} = -\operatorname{grad} \Phi + \operatorname{rot} (\Psi \mathfrak{k}) \tag{7}$$

zerlegen in den negativen Gradienten eines (einwertigen) skalaren Potentials Φ *und in die Rotation eines (einwertigen) Vektorpotentials* $\Psi \mathfrak{k}$*; dabei sind*

$$\Phi = -\frac{1}{2\pi} \sum_{i=1}^{p} \int_{\mathfrak{S}_i} v_n F \, ds, \tag{8}$$

$$\Psi = -\frac{1}{2\pi} \sum_{i=1}^{p} \int_{\mathfrak{S}_i} v_t F \, ds \tag{9}$$

in $\mathfrak{G}$ *beliebig oft stetig differenzierbare und harmonische Funktionen und* $v_n = (\mathfrak{n}, \mathfrak{v})$, $v_t = (\mathfrak{t}, \mathfrak{v})$ *die Randkomponenten von* $\mathfrak{v}$.

Beweis. Überführt man das Gebiet $\mathfrak{G}$ im Falle $p > 1$ durch $p-1$ Schnitte in ein einfach zusammenhängendes, so ist die Behauptung für $p > 1$ sicher richtig, wenn sie für $p = 1$ gilt, da die über die Schnittberandungen erstreckten Integrale gemeinsam keinen Beitrag liefern. Es genügt daher, den Beweis für ein einfach zusammenhängendes Gebiet $\mathfrak{G}$ mit stückweise glatter Berandung $\mathfrak{S}$ zu erbringen. Aus der einmaligen stetigen Differenzierbarkeit von $\mathfrak{v}$ in $\mathfrak{G}$ und der Divergenz- und Rotationsfreiheit folgt die Existenz von stetigen Ableitungen beliebig hoher Ordnung aus der Tatsache, daß $w = v_x - i v_y$ eine analytische Funktion ist, da $\operatorname{div} \mathfrak{v} = 0$ und $\operatorname{rot}_z \mathfrak{v} = 0$ mit den Cauchy-Riemannschen Differentialgleichungen identisch sind. Außerdem sind Real- und Imaginärteil von w und damit die Komponenten von $\mathfrak{v}$ in $\mathfrak{G}$ harmonische Funktionen. Der in $\mathfrak{G}$ liegende Aufpunkt (x, y) werde von einem hinreichend kleinen Kreis $\mathfrak{K}$ umschlossen. Dann sei $\mathfrak{S}^*$ eine in $\mathfrak{G}$ verlaufende, stückweise glatte, doppelpunktfreie, geschlossene Kurve, die $\mathfrak{K}$ umschließt, ohne mit $\mathfrak{K}$ einen gemeinsamen Punkt zu besitzen[1]. Durch Anwendung des Greenschen Satzes auf das von $\mathfrak{S}^*$ und $\mathfrak{K}$ berandete zweifach zusammenhängende Gebiet $\mathfrak{G}^*$ folgt

$$\iint_{\mathfrak{G}^*} (\mathfrak{v}\, \Delta F - F \Delta \mathfrak{v})\, d\xi\, d\eta = \int_{\mathfrak{S}^*} \left(\mathfrak{v} \frac{\partial F}{\partial n} - F \frac{\partial \mathfrak{v}}{\partial n} \right) ds - \int_{\mathfrak{K}} \left(\mathfrak{v} \frac{\partial F}{\partial n} - F \frac{\partial \mathfrak{v}}{\partial n} \right) ds. \tag{10}$$

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[1] Bezüglich der Orientierung von Normale und Tangente auf $\mathfrak{S}$, $\mathfrak{K}$ und $\mathfrak{S}^*$ gilt das in § 2 Gesagte.

Zusammenfassung von (1) und (2) in

$$\operatorname{grad} v_x = [\mathfrak{k}, \operatorname{grad} v_y]$$

und skalare Multiplikation mit $\mathfrak{n}$ und $\mathfrak{t}$ ergibt

$$\frac{\partial v_x}{\partial n} = -\frac{\partial v_y}{\partial t}, \qquad \frac{\partial v_x}{\partial t} = \frac{\partial v_y}{\partial n}$$

oder

$$\frac{\partial \mathfrak{v}}{\partial n} = \frac{\partial [\mathfrak{k}, \mathfrak{v}]}{\partial t}. \tag{11}$$

Aus (10), (11), $\Delta F = 0$ und $\Delta \mathfrak{v} = 0$ folgt nach Ausführung einer partiellen Integration

$$\int\limits_{\mathfrak{S}^*} \left(\mathfrak{v} \frac{\partial F}{\partial n} + [\mathfrak{k}, \mathfrak{v}] \frac{\partial F}{\partial t}\right) ds - \int\limits_{\mathfrak{K}} \left(\mathfrak{v} \frac{\partial F}{\partial n} + [\mathfrak{k}, \mathfrak{v}] \frac{\partial F}{\partial t}\right) ds = 0.$$

Mittels der (durch skalare Multiplikation mit $\mathfrak{n}$ und $\mathfrak{t}$ unmittelbar zu beweisenden) Identität

$$\mathfrak{v} \frac{\partial F}{\partial n} + [\mathfrak{k}, \mathfrak{v}] \frac{\partial F}{\partial t} = v_n \operatorname{grad} F - v_t [\operatorname{grad} F, \mathfrak{k}]$$

und der auf $\mathfrak{K}$ gültigen Gleichungen

$$\frac{\partial F}{\partial n} = -\frac{1}{r} + \frac{\partial F'}{\partial n}, \qquad \frac{\partial F}{\partial t} = \frac{\partial F'}{\partial t}$$

ergibt sich

$$\int\limits_{\mathfrak{K}} \frac{\mathfrak{v}}{r} ds - \int\limits_{\mathfrak{K}} \left(\mathfrak{v} \frac{\partial F'}{\partial n} + [\mathfrak{k}, \mathfrak{v}] \frac{\partial F'}{\partial t}\right) ds$$
$$= -\int\limits_{\mathfrak{S}^*} v_n \operatorname{grad} F\, ds + \left[\int\limits_{\mathfrak{S}^*} v_t \operatorname{grad} F\, ds, \mathfrak{k}\right].$$

Strebt der Radius des Kreises $\mathfrak{K}$ gegen Null, so konvergieren die Integrale der linken Seite bzw. gegen $2\pi\mathfrak{v}$ und Null. Denkt man jetzt den Gradienten auf den Aufpunkt statt auf den Integrationspunkt angewandt (es tritt dann wegen $F(x, y; \xi, \eta) = f(x - \xi, y - \eta)$ ein Vorzeichenwechsel ein) und jeweils vor das Integralzeichen gezogen, so bekommt man

$$2\pi\mathfrak{v} = \operatorname{grad} \int\limits_{\mathfrak{S}^*} v_n F\, ds - \left[\operatorname{grad} \int\limits_{\mathfrak{S}^*} v_t F\, ds, \mathfrak{k}\right].$$

Die rechte Seite bleibt ungeändert, wenn man den Integrationsweg $\mathfrak{S}^*$ in $\mathfrak{S}$ überführt. Mit Hilfe der Identität

$$\operatorname{rot}(\Psi\mathfrak{k}) = [\operatorname{grad} \Psi, \mathfrak{k}] \tag{12}$$

ergibt sich dann die Behauptung (7) bis (9).

Satz 1. *Ein Vektorfeld $\mathfrak{v}$ mit den in § 2 genannten Eigenschaften ist überall außerhalb $\mathfrak{C}$ vollständig bestimmt durch seine Werte auf $\mathfrak{C}$ und im Unendlichen und wird dargestellt durch*

$$\mathfrak{v} = \mathfrak{v}_\infty - \operatorname{grad} \Phi + \operatorname{rot}(\Psi\mathfrak{k}) \tag{13}$$

mit

$$\Phi = \frac{1}{2\pi} \int\limits_{\mathfrak{C}} v_n \ln \frac{1}{r}\, ds, \tag{14}$$

$$\Psi = \frac{1}{2\pi} \int\limits_{\mathfrak{C}} v_t \ln \frac{1}{r}\, ds. \tag{15}$$

Beweis. Umgibt man den (außerhalb $\mathfrak{C}$ liegenden) Aufpunkt (x, y) mit einem hinreichend großen Kreis $\mathfrak{S}$ vom Radius R derart, daß $\mathfrak{C}$ ganz im Inneren von $\mathfrak{S}$ liegt, so gilt, wenn man $F = \ln \frac{1}{r}$ setzt, auf Grund des Lemmas für das von $\mathfrak{S}$ und $\mathfrak{C}$ berandete, zweifach zusammenhängende Gebiet

$$\mathfrak{v} = -\operatorname{grad}(\Phi^* + \Phi) + \operatorname{rot}\big((\Psi^* + \Psi)\,\mathfrak{k}\big)$$

mit

$$\Phi^* = -\frac{1}{2\pi} \int\limits_{\mathfrak{S}} v_n \ln \frac{1}{r}\, ds, \qquad \Phi = \frac{1}{2\pi} \int\limits_{\mathfrak{C}} v_n \ln \frac{1}{r}\, ds,$$

$$\Psi^* = -\frac{1}{2\pi} \int\limits_{\mathfrak{S}} v_t \ln \frac{1}{r}\, ds, \qquad \Psi = \frac{1}{2\pi} \int\limits_{\mathfrak{C}} v_t \ln \frac{1}{r}\, ds.$$

Es genügt dann bei Beachtung von (12) offenbar, die Gültigkeit der Gleichung

$$-\operatorname{grad} \Phi^* + [\operatorname{grad} \Psi^*, \mathfrak{k}] = \mathfrak{v}_\infty \tag{16}$$

im Aufpunkt (x, y) zu bestätigen. Zieht man die Gradienten unter die Integralzeichen, so gilt im Punkte (x, y)

$$\operatorname{grad} \Phi^* = -\frac{1}{2\pi} \int\limits_{\mathfrak{S}} v_n \frac{\mathfrak{n}}{R}\, ds,$$

$$\operatorname{grad} \Psi^* = -\frac{1}{2\pi} \int\limits_{\mathfrak{S}} v_t \frac{\mathfrak{n}}{R}\, ds;$$

folglich:

$$-\operatorname{grad} \Phi^* + [\operatorname{grad} \Psi^*, \mathfrak{k}] = \frac{1}{2\pi R} \int\limits_{\mathfrak{S}} (v_n \mathfrak{n} + v_t \mathfrak{t})\, ds = \frac{1}{2\pi R} \int\limits_{\mathfrak{S}} \mathfrak{v}\, ds. \tag{17}$$

Die beiden für alle $0 \leqq \Theta \leqq 2\pi$ im Sinne gleichmäßiger Konvergenz existierenden Grenzwerte

$$\lim_{R\to\infty} \mathfrak{v}(x + R\cos\Theta,\, y + R\sin\Theta) = \mathfrak{v}_\infty \tag{18}$$

sind eine unmittelbare Folge der Voraussetzung (3): Sei nämlich $R' > 0$ der Radius eines Kreises um den Ursprung derart, daß zu vorgegebenem $\varepsilon > 0$ für alle $R \geqq R'$ und alle $0 \leqq \Theta \leqq 2\pi$

$$|v_x(R\cos\Theta, R\sin\Theta) - v_{\infty,x}| < \varepsilon,$$

$$|v_y(R\cos\Theta, R\sin\Theta) - v_{\infty,y}| < \varepsilon$$

wird, und $R'' \geqq R'$ der Radius eines Kreises um (x, y), der den zuerst genannnten Kreis umschließt, so gilt auch für alle $R \geqq R''$ und alle $0 \leqq \Theta \leqq 2\pi$

$$|v_x(x + R\cos\Theta,\ y + R\sin\Theta) - v_{\infty,x}| < \varepsilon,$$

$$|v_y(x + R\cos\Theta,\ y + R\sin\Theta) - v_{\infty,y}| < \varepsilon.$$

Es folgt daher mit (18)

$$\lim_{R\to\infty} \frac{1}{2\pi R} \int\limits_{\mathfrak{S}} \mathfrak{v}\, ds = \lim_{R\to\infty} \frac{1}{2\pi} \int\limits_0^{2\pi} \mathfrak{v}(x + R\cos\Theta, y + R\sin\Theta)\, d\Theta = \mathfrak{v}_\infty,$$

und der in (17) durchgeführte Grenzübergang $R \to \infty$ liefert (16).

Im Hinblick auf das Folgende betrachten wir in irgendeinem außerhalb $\mathfrak{C}$ gelegenen Punkt (x, y) zwei Einheitsvektoren $\mathfrak{n}^*$ und $\mathfrak{t}^*$ mit der Maßgabe, daß $\mathfrak{t}^*$ aus $\mathfrak{n}^*$ durch Drehung um $+\pi/2$ hervorgeht. Dann folgt aus (13) in Verbindung mit (12) durch skalare Multiplikation

$$(\mathfrak{n}^*, \mathfrak{v}) = (\mathfrak{n}^*, \mathfrak{v}_\infty) - \frac{\partial \Phi}{\partial n^*} + \frac{\partial \Psi}{\partial t^*}, \tag{19}$$

$$(\mathfrak{t}^*, \mathfrak{v}) = (\mathfrak{t}^*, \mathfrak{v}_\infty) - \frac{\partial \Phi}{\partial t^*} - \frac{\partial \Psi}{\partial n^*}. \tag{20}$$

Speziell erhält man für $\mathfrak{n}^* = \mathfrak{i}$ und $\mathfrak{t}^* = \mathfrak{j}$ die Komponentenzerlegung von (13)

$$v_x = v_{\infty,x} - \frac{\partial \Phi}{\partial x} + \frac{\partial \Psi}{\partial y}, \tag{21}$$

$$v_y = v_{\infty,y} - \frac{\partial \Phi}{\partial y} - \frac{\partial \Psi}{\partial x}. \tag{22}$$

§ 4. Äquivalenzsatz

Mit der vollständigen Beherrschung des Vektorfeldes $\mathfrak{v}$ durch seine Randwerte auf $\mathfrak{C}$ und im Unendlichen ist freilich nicht gesagt, daß diese Daten beliebig vorgebbar sind. Vielmehr stellen wir gerade die Aufgabe, zu festem $\mathfrak{v}_\infty$ alle miteinander „verträglichen" Randkomponenten v_n, v_t auf $\mathfrak{C}$ zu ermitteln. Dies führt im folgenden auf die Auflösung von linearen Integralgleichungen.

Satz 2. *Es seien v_n und v_t zwei auf $\mathfrak{C}$ stetige Funktionen. Dann ist das Bestehen irgendeiner der beiden Integralgleichungen*

$$v_n + \frac{1}{\pi} \frac{\partial}{\partial n} \int\limits_{\mathfrak{C}} v_n \ln \frac{1}{r}\, ds = 2(\mathfrak{n}, \mathfrak{v}_\infty) + \frac{1}{\pi} \frac{\partial}{\partial t} \int\limits_{\mathfrak{C}} v_t \ln \frac{1}{r}\, ds, \tag{23}$$

$$v_t + \frac{1}{\pi} \frac{\partial}{\partial n} \int\limits_{\mathfrak{C}} v_t \ln \frac{1}{r}\, ds = 2(\mathfrak{t}, \mathfrak{v}_\infty) - \frac{1}{\pi} \frac{\partial}{\partial t} \int\limits_{\mathfrak{C}} v_n \ln \frac{1}{r}\, ds \tag{24}$$

notwendig und hinreichend für die Existenz eines Vektorfeldes $\mathfrak{v}$, das außer den in § 2 genannten Eigenschaften auf $\mathfrak{C}$ die Normalkomponente v_n und die Tangentialkomponente v_t besitzt. Zu jeder Kombination v_n, v_t, die (23) *oder* (24) *gehorcht, gibt es genau ein derartiges Feld $\mathfrak{v}$; dieses wird durch die Gleichungen* (13) *bis* (15) *dargestellt.*

Beweis. Sei $\mathfrak{C}^*$ eine äußere äquidistante Nachbarkurve[1] zu $\mathfrak{C}$ und seien $\mathfrak{n}^*$ und $\mathfrak{t}^*$ Einheitsnormale und -tangente auf $\mathfrak{C}^*$. Dann gilt für das auf $\mathfrak{C}^*$ genommene Potential einer stetigen einfachen Belegung ϱ auf $\mathfrak{C}$ bei Annäherung von $\mathfrak{C}^*$ an die Außenseite von $\mathfrak{C}$, die mit $\mathfrak{C}_a$ bezeichnet werde,

$$\lim_{\mathfrak{C}^* \to \mathfrak{C}_a} \frac{\partial}{\partial n^*} \int_{\mathfrak{C}} \varrho \ln \frac{1}{r}\, ds = \frac{\partial}{\partial n} \int_{\mathfrak{C}} \varrho \ln \frac{1}{r}\, ds - \pi \varrho, \tag{25}$$

$$\lim_{\mathfrak{C}^* \to \mathfrak{C}_a} \frac{\partial}{\partial t^*} \int_{\mathfrak{C}} \varrho \ln \frac{1}{r}\, ds = \frac{\partial}{\partial t} \int_{\mathfrak{C}} \varrho \ln \frac{1}{r}\, ds; \tag{26}$$

dabei sind die auf den rechten Seiten auftretenden Differentiationen unter den Integralzeichen ausgeführt zu denken und die Integrale selbst als Cauchysche Hauptwerte zu verstehen [*3*, *4*, *5*]. Beachtet man, daß Φ und Ψ in (14) und (15) als Potentiale der einfachen Belegungen $v_n/2\pi$ und $v_t/2\pi$ dargestellt sind, so gehen (19) und (20) in die mit (23) und (24) identischen Gleichungen

$$v_n = (\mathfrak{n}, \mathfrak{v}_\infty) - \frac{1}{2\pi} \frac{\partial}{\partial n} \int_{\mathfrak{C}} v_n \ln \frac{1}{r}\, ds + \frac{v_n}{2} + \frac{1}{2\pi} \frac{\partial}{\partial t} \int_{\mathfrak{C}} v_t \ln \frac{1}{r}\, ds,$$

$$v_t = (\mathfrak{t}, \mathfrak{v}_\infty) - \frac{1}{2\pi} \frac{\partial}{\partial t} \int_{\mathfrak{C}} v_n \ln \frac{1}{r}\, ds - \frac{1}{2\pi} \frac{\partial}{\partial n} \int_{\mathfrak{C}} v_t \ln \frac{1}{r}\, ds + \frac{v_t}{2}$$

über, wenn sich der Aufpunkt (x, y) unbegrenzt dem Profil nähert. Für den Beweis der Umkehrung erklären wir durch (13) bis (15) ein überall außerhalb $\mathfrak{C}$ stetig differenzierbares und harmonisches Vektorfeld $\mathfrak{v}$, wobei die v_n und v_t auf $\mathfrak{C}$ stetig sind und irgendeiner der beiden Integralgleichungen (23) und (24) gehorchen. Da die Grenzwerte (25) und (26) für stetig gekrümmte Profile und stetige Belegungen ϱ im Sinne gleichmäßiger Konvergenz existieren [*5*], können wir $\mathfrak{v}$ überall außerhalb $\mathfrak{C}$ und auf $\mathfrak{C}_a$ als stetiges Vektorfeld erklären. Die Ableitungen von Φ und Ψ verschwinden im Unendlichen, so daß $\mathfrak{v}$ dort gegen $\mathfrak{v}_\infty$ strebt. Wir zeigen, daß diese Konvergenz eine gleichmäßige ist. Sei (ξ, η) ein laufender Punkt auf $\mathfrak{C}$ und U der Umfang von $\mathfrak{C}$, so gibt es Schranken $M > 0$ und $M^* > 0$, mit denen auf $\mathfrak{C}$

$$|\xi| + |\eta| \leqq \frac{M}{2},$$

$$|v_n| \leqq \frac{M^*}{U}, \qquad |v_t| \leqq \frac{M^*}{U}$$

wird. Zu vorgegebenem $\varepsilon > 0$ gibt es dann immer ein $R' > M$ derart, daß für alle $R \geqq R'$

$$\frac{M^*}{R} \, \frac{R+M}{R-M} < \varepsilon$$

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[1] Eine zu einer gegebenen Kurve $\mathfrak{C}$ äquidistante Nachbarkurve $\mathfrak{C}^*$ wird definiert durch ihren Ortsvektor

$$\mathfrak{x}^* = \mathfrak{x} + \delta\, \mathfrak{n},$$

wobei $\mathfrak{x}$ und $\mathfrak{n}$ bzw. Ortsvektor und Normale von $\mathfrak{C}$ sind und δ eine von Null verschiedene, dem Betrage nach hinreichend kleine Zahl ist. Aus der geometrisch unmittelbar einleuchtenden Tatsache, daß entsprechende Punkte von $\mathfrak{C}$ und $\mathfrak{C}^*$ einen gemeinsamen Krümmungsmittelpunkt besitzen, folgt, daß sich die Krümmungsradien genau um δ unterscheiden und daß somit $\mathfrak{C}^*$ eine stetige Krümmung besitzt, wenn dies für $\mathfrak{C}$ der Fall ist.

gilt. Für einen beliebigen Aufpunkt $(R\cos\Theta, R\sin\Theta)$ mit $R \geqq R'$ und $0 \leqq \Theta \leqq 2\pi$ folgt dann aus (14)

$$\frac{\partial\Phi}{\partial x} = -\frac{1}{2\pi}\int\limits_{\mathfrak{C}} v_n \frac{R\cos\Theta - \xi}{R^2 - 2R(\xi\cos\Theta + \eta\sin\Theta) + \xi^2 + \eta^2}\, ds,$$

$$\left|\frac{\partial\Phi}{\partial x}\right| \leqq \frac{1}{2\pi}\int\limits_{\mathfrak{C}} \frac{M^*}{U}\,\frac{R+M}{R^2 - RM}\, ds$$

$$= \frac{M^*}{2\pi R}\,\frac{R+M}{R-M} < \frac{\varepsilon}{2\pi};$$

da sich die Ableitungen $\partial\Phi/\partial y$, $\partial\Psi/\partial x$, $\partial\Psi/\partial y$ entsprechend abschätzen lassen, liefern (12) und (13)

$$|\mathfrak{v} - \mathfrak{v}_\infty| \leqq |\operatorname{grad}\Phi| + |\operatorname{grad}\Psi| \leqq \frac{\sqrt{2}}{\pi}\,\varepsilon < \varepsilon,$$

also auch

$$|v_x - v_{\infty,x}| \leqq |\mathfrak{v} - \mathfrak{v}_\infty| < \varepsilon,$$

$$|v_y - v_{\infty,y}| \leqq |\mathfrak{v} - \mathfrak{v}_\infty| < \varepsilon.$$

Damit hat $\mathfrak{v}$ die in § 2 genannten Eigenschaften. Es bleibt zu zeigen, daß $\mathfrak{v}$ auf $\mathfrak{C}_a$ die Komponenten v_n und v_t besitzt. Bei Annäherung des Aufpunktes an $\mathfrak{C}$ liefern (19) und (20) bei Beachtung von (25) und (26)

$$(27)\quad \lim_{\mathfrak{C}^*\to\mathfrak{C}_a}(\mathfrak{n}^*, \mathfrak{v}) = (\mathfrak{n}, \mathfrak{v}_\infty) - \frac{1}{2\pi}\frac{\partial}{\partial n}\int\limits_{\mathfrak{C}} v_n \ln\frac{1}{r}\, ds + \frac{v_n}{2} + \frac{1}{2\pi}\frac{\partial}{\partial t}\int\limits_{\mathfrak{C}} v_t \ln\frac{1}{r}\, ds,$$

$$(28)\quad \lim_{\mathfrak{C}^*\to\mathfrak{C}_a}(\mathfrak{t}^*, \mathfrak{v}) = (\mathfrak{t}, \mathfrak{v}_\infty) - \frac{1}{2\pi}\frac{\partial}{\partial t}\int\limits_{\mathfrak{C}} v_n \ln\frac{1}{r}\, ds - \frac{1}{2\pi}\frac{\partial}{\partial n}\int\limits_{\mathfrak{C}} v_t \ln\frac{1}{r}\, ds + \frac{v_t}{2}.$$

Genügen nun v_n und v_t der Integralgleichung (23), so liefert (27)

$$(29)\qquad \lim_{\mathfrak{C}^*\to\mathfrak{C}_a}(\mathfrak{n}^*, \mathfrak{v}) = v_n;$$

ist dagegen (24) erfüllt, so läßt sich mit (28) der Schluß

$$(30)\qquad \lim_{\mathfrak{C}^*\to\mathfrak{C}_a}(\mathfrak{t}^*, \mathfrak{v}) = v_t$$

ziehen. Denken wir uns $\mathfrak{v}$ durch (13) bis (15) auch im Innengebiet $\mathfrak{B}$ und auf der Innenseite $\mathfrak{C}_i$ von $\mathfrak{C}$ als stetiges Vektorfeld erklärt und beachten dabei außer (25) und (26) noch die analogen Beziehungen für die Innenseite [*3, 4, 5*]

$$(31)\quad \lim_{\mathfrak{C}^{**}\to\mathfrak{C}_i}\frac{\partial}{\partial n^{**}}\int\limits_{\mathfrak{C}} \varrho \ln\frac{1}{r}\, ds = \frac{\partial}{\partial n}\int\limits_{\mathfrak{C}} \varrho \ln\frac{1}{r}\, ds + \pi\varrho,$$

$$(32)\quad \lim_{\mathfrak{C}^{**}\to\mathfrak{C}_i}\frac{\partial}{\partial t^{**}}\int\limits_{\mathfrak{C}} \varrho \ln\frac{1}{r}\, ds = \frac{\partial}{\partial t}\int\limits_{\mathfrak{C}} \varrho \ln\frac{1}{r}\, ds,$$

wobei $\mathfrak{C}^{**}$ eine innere äquidistante Nachbarkurve zu $\mathfrak{C}$ mit der Normalen $\mathfrak{n}^{**}$ und der Tangente $\mathfrak{t}^{**}$ ist, so erhalten wir mit Hilfe von (19) und (20) den Sprung, den $\mathfrak{v}$ beim Durchgang durch $\mathfrak{C}$ erleidet:

$$(33)\qquad \lim_{\mathfrak{C}^*\to\mathfrak{C}_a}(\mathfrak{n}^*, \mathfrak{v}) - \lim_{\mathfrak{C}^{**}\to\mathfrak{C}_i}(\mathfrak{n}^{**}, \mathfrak{v}) = v_n,$$

$$(34)\qquad \lim_{\mathfrak{C}^*\to\mathfrak{C}_a}(\mathfrak{t}^*, \mathfrak{v}) - \lim_{\mathfrak{C}^{**}\to\mathfrak{C}_i}(\mathfrak{t}^{**}, \mathfrak{v}) = v\,.$$

Je nachdem, ob nun (29) oder (30) erfüllt ist, liefert entweder (33)

$$\lim_{\mathfrak{C}^{**}\to\mathfrak{C}_i}(\mathfrak{n}^{**},\mathfrak{v})=0 \tag{35}$$

oder (34)

$$\lim_{\mathfrak{C}^{**}\to\mathfrak{C}_i}(\mathfrak{t}^{**},\mathfrak{v})=0. \tag{36}$$

Liege (x_0, y_0) auf $\mathfrak{C}_i$ und (x, y) in $\mathfrak{B}$ oder auf $\mathfrak{C}_i$, so definiert das vom Wege unabhängige Integral

$$u=\int_{x_0,y_0}^{x,y} v_x\,dx+v_y\,dy \tag{37}$$

eine in $\mathfrak{B}$ und auf $\mathfrak{C}_i$ stetig differenzierbare Funktion $u(x, y)$ mit

$$\operatorname{grad} u=\mathfrak{v},$$

die überdies in $\mathfrak{B}$ beliebig oft stetig differenzierbar und harmonisch ist und im Falle der Gültigkeit von (35) die Eigenschaft

$$\lim_{\mathfrak{C}^{**}\to\mathfrak{C}_i}\frac{\partial u}{\partial n^{**}}=0, \tag{38}$$

besitzt; anderenfalls ergibt (36) in Verbindung mit (37)

$$\lim_{\mathfrak{C}^{**}\to\mathfrak{C}_i} u=0 \tag{39}$$

(man braucht dazu nur den Integrationsweg von (37) entlang der Berandung $\mathfrak{C}_i$ zu führen). Anwendung des Greenschen Satzes auf das von $\mathfrak{C}^{**}$ umschlossene Gebiet liefert

$$\iint(u\,\Delta u+\operatorname{grad}^2 u)\,dx\,dy=\int_{\mathfrak{C}^{**}} u\,\frac{\partial u}{\partial n^{**}}\,ds$$

und damit durch Grenzübergang $\mathfrak{C}^{**}\to\mathfrak{C}_i$ in jedem der beiden Fälle (38) und (39)

$$\iint_{\mathfrak{B}}\mathfrak{v}^2\,dx\,dy=0.$$

Also verschwindet $\mathfrak{v}$ in $\mathfrak{B}$ und auf $\mathfrak{C}$, und (33) und (34) besagen, daß das Vektorfeld des Außengebiets auf $\mathfrak{C}_a$ die vorgegebenen Komponenten v_n und v_t besitzt. Zum Beweis der Eindeutigkeit nehmen wir an, es seien $\mathfrak{v}_1$ und $\mathfrak{v}_2$ zwei Vektorfelder mit den geforderten Eigenschaften. Da diese Felder die gleichen Randwerte v_n, v_t, $\mathfrak{v}_\infty$ besitzen, sind sie beide durch (13) bis (15) darstellbar und somit identisch.

Ist eine der beiden Komponenten v_n und v_t vorgeschrieben, so kann man zur Bestimmung der anderen grundsätzlich zwischen (23) und (24), d.h. zwischen einer Integralgleichung erster und einer solchen zweiter Art wählen. Auf Grund der bekannten Tatsache, daß der Kern $\frac{\partial}{\partial n}\ln\frac{1}{r}$ stetig, dagegen $\frac{\partial}{\partial t}\ln\frac{1}{r}$ singulär ist, hat man hiermit die Wahl zwischen einer Fredholmschen Integralgleichung zweiter Art mit stetigem Kern und einer Integralgleichung erster Art mit singulärem Kern. Indem wir uns anschließend für den ersten Weg entscheiden, d.h. v_n bei vorgegebenem v_t aus (23) und v_t bei vorgegebenem v_n aus (24) berechnen, stehen uns für Fragen der Theorie die Fredholmschen Sätze und im Hinblick auf praktische Anwendungen einfache numerische Verfahren zur Verfügung.

§ 5. Allgemeine Lösungen der Integralgleichungen zweiter Art

Die Integralgleichungen (23) und (24) sind von der Form[1]

$$\sigma + \frac{1}{\pi} \frac{\partial}{\partial n} \int\limits_{\mathfrak{C}} \sigma \ln \frac{1}{r}\, ds = \tau . \tag{40}$$

Man weiß, daß die zu (40) transponierte homogene Integralgleichung als allgemeine Lösung eine beliebige Konstante besitzt [*4*, *6*]. Infolgedessen hat (40) nach den Fredholmschen Sätzen [*3*, *4*, *6*] dann und nur dann eine auf $\mathfrak{C}$ stetige Lösung σ, wenn τ stetig ist und zur allgemeinen Lösung der transponierten homogenen Gleichung orthogonal ist, d.h. der Bedingung

$$\int\limits_{\mathfrak{C}} \tau\, ds = 0 \tag{41}$$

genügt.

Um diese Bedingungen im Falle der Fredholmschen Integralgleichungen (23) und (24) nachzuprüfen, genügt es offenbar, da $(\mathfrak{n}, \mathfrak{v}_\infty)$ und $(\mathfrak{t}, \mathfrak{v}_\infty)$ auf $\mathfrak{C}$ stetig sind und im Mittel verschwinden, die zu einer auf $\mathfrak{C}$ stetigen Funktion ϱ ,,konjugierte“ Funktion (Cauchyscher Hauptwert!)

$$k = \frac{1}{\pi} \frac{\partial}{\partial t} \int\limits_{\mathfrak{C}} \varrho \ln \frac{1}{r}\, ds \tag{42}$$

auf ihr Verschwinden im Mittel hin zu untersuchen. Zu diesem Zweck werde eine gewisse (äußere) Nachbarschaft von $\mathfrak{C}$ mit einer Schar äquidistanter, stetig gekrümmter Kurven $\mathfrak{C}_\delta^*$ schlicht überdeckt. Der Index δ gebe dabei den Abstand zwischen $\mathfrak{C}_\delta^*$ und $\mathfrak{C}$ an. Die Orthogonaltrajektorien dieser Schar, die gerade Linien sind, gestatten es, auf den $\mathfrak{C}_\delta^*$ einen Kurvenparameter s einzuführen, der der im Gegenuhrzeigersinn wachsenden Bogenlänge an der entsprechenden Stelle von $\mathfrak{C}$ entspricht, jedoch im allgemeinen nicht mit der (ebenfalls im Gegenuhrzeigersinn wachsenden) Bogenlänge s^* auf $\mathfrak{C}_\delta^*$ identisch ist. Damit haben wir in einer (äußeren) Umgebung von $\mathfrak{C}$ ein krummliniges, orthogonales Kurvennetz (δ, s) erklärt, das $\mathfrak{C}$ als Parameterlinie $\delta = 0$ enthält. Sei $\partial/\partial t^*$ die tangentiale Differentiation auf $\mathfrak{C}_\delta^*$, so ist

$$k^* = \frac{1}{\pi} \frac{\partial}{\partial t^*} \int\limits_{\mathfrak{C}} \varrho \ln \frac{1}{r}\, ds$$

eine stetige Funktion von δ und s für $\delta > 0$. Da überdies k^* gemäß (26) für $\delta \to 0$ gleichmäßig für alle s gegen den durch (42) gegebenen Grenzwert k strebt, ist dieser selbst auf $\mathfrak{C}$ stetig. Bezeichnet ds/ds^* das Verhältnis der Bogenlängen von $\mathfrak{C}$ und $\mathfrak{C}_\delta^*$ an Stellen mit gleichem Kurvenparameter s, so folgt

$$k^* = \frac{1}{\pi} \frac{ds}{ds^*} \frac{\partial}{\partial s} \int\limits_{\mathfrak{C}} \varrho \ln \frac{1}{r}\, ds . \tag{43}$$

---

[1] Auf diese Integralgleichung wird man z.B. auch bei der Behandlung des inneren Neumannschen Problems geführt, wo man nach in $\mathfrak{B}$ harmonischen Funktionen sucht, die auf $\mathfrak{C}$ die Normalableitung $\pi\tau$ besitzen. Die Lösungen dieses Problems werden durch alle Potentiale von einfachen Belegungen σ dargestellt, die der Integralgleichung (40) genügen [*4*, *5*, *6*].

Nun ist ds/ds^* eine stetig differenzierbare Funktion von δ und s, da alle Parameterlinien des (δ, s)-Netzes stetig gekrümmt sind. Integrieren wir (43) für festes $\delta > 0$ über das Intervall $0 \leqq s \leqq U$, wobei U der Umfang von $\mathfrak{C}$ ist, so wird daher nach einer partiellen Integration

$$\int\limits_0^U k^* \, ds = -\frac{1}{\pi} \int\limits_0^U \left(\frac{\partial}{\partial s} \frac{ds}{ds^*}\right) \left(\int\limits_{\mathfrak{C}} \varrho \ln \frac{1}{r} \, ds\right) ds. \tag{44}$$

Hier konvergieren außer k^* auch die Faktoren des rechts stehenden Integranden, insbesondere das (gleichmäßig stetige) Potential einer einfachen Belegung [5], gleichmäßig für alle $0 \leqq s \leqq U$, wenn $\delta \to 0$ strebt; da ds/ds^* auf $\mathfrak{C}$ den Wert 1 besitzt, gilt

$$\lim_{\delta \to 0} \frac{\partial}{\partial s} \frac{ds}{ds^*} = 0,$$

so daß der Integrand rechts in (44) gleichmäßig gegen Null konvergiert. Wir dürfen daher in (44) die Grenzübergänge $\delta \to 0$ mit den Integrationen vertauschen und erhalten somit das gewünschte Resultat

$$\int\limits_{\mathfrak{C}} k \, ds = 0. \tag{45}$$

Die Existenz einer Lösung für die Integralgleichungen (23) und (24) bzw. für (40) ist damit gesichert. Die allgemeine Lösung setzt sich dann bekanntlich aus der allgemeinen Lösung der homogenen und einer speziellen Lösung der inhomogenen Gleichung zusammen. Die Mannigfaltigkeit der Lösungen $\overset{0}{\sigma}$ der homogenen Gleichung

$$\overset{0}{\sigma} + \frac{1}{\pi} \frac{\partial}{\partial n} \int\limits_{\mathfrak{C}} \overset{0}{\sigma} \ln \frac{1}{r} \, ds = 0 \tag{46}$$

entspricht nach den Fredholmschen Sätzen der der transponierten homogenen Gleichung, sie geht also aus einer nichttrivialen Lösung von (46) durch Multiplikation mit einem beliebigen konstanten Faktor hervor. Weiter gilt, daß eine im Mittel verschwindende Lösung von (46) notwendig die triviale ist. Infolgedessen gibt es genau eine durch

$$\int\limits_{\mathfrak{C}} \overset{0}{\sigma} \, ds = 1 \tag{47}$$

normierte Lösung $\overset{0}{\sigma}$ von (46)[1]. Aus dem gleichen Grunde existiert genau eine spezielle Lösung $\bar{\sigma}$ der inhomogenen Gleichung (40), die der Nebenbedingung

$$\int\limits_{\mathfrak{C}} \bar{\sigma} \, ds = 0 \tag{48}$$

gehorcht. Die allgemeine Lösung von (40) lautet damit

$$\sigma = \text{const} \, \overset{0}{\sigma} + \bar{\sigma}. \tag{49}$$

---

[1] Im folgenden verstehen wir unter $\overset{0}{\sigma}$ die einzige Lösung von (46) und (47); die allgemeine Lösung von (46) lautet dann const $\overset{0}{\sigma}$.

Bezeichnen k_n und k_t die durch (42) definierten Konjugierten zu v_n und v und bedeutet ein Akzent die Differentiation nach der im Gegenuhrzeigersinn wachsenden Bogenlänge s, so lauten (23) und (24)

$$v_n + \frac{1}{\pi}\frac{\partial}{\partial n}\int_{\mathfrak{C}} v_n \ln\frac{1}{r}\,ds = 2v_{\infty,x}\,y' - 2v_{\infty,y}\,x' + k_t, \tag{50}$$

$$v_t + \frac{1}{\pi}\frac{\partial}{\partial n}\int_{\mathfrak{C}} v_t \ln\frac{1}{r}\,ds = 2v_{\infty,x}\,x' + 2v_{\infty,y}\,y' - k_n. \tag{51}$$

Die im Mittel verschwindenden speziellen Lösungen der inhomogenen Gleichungen (50) und (51) lassen sich jeweils in drei im Mittel verschwindende Bestandteile zerlegen, die bzw. von den drei Summanden der rechten Seiten herrühren. Da die allgemeine Lösung die Form (49) besitzt, werden alle Normalkomponenten im Falle einer vorgegebenen Tangentialkomponente dargestellt durch

$$v_n = E\,\overset{0}{v}_n + v_{\infty,x}\overset{1}{v}_n + v_{\infty,y}\overset{2}{v}_n + \overset{3}{v}_n, \tag{52}$$

wobei die „Grundlösungen“ des Problems $\overset{0}{v}_n$, $\overset{1}{v}_n$, $\overset{2}{v}_n$, $\overset{3}{v}_n$ eindeutig bestimmt sind durch

$$\begin{aligned}
&\overset{0}{v}_n + \frac{1}{\pi}\frac{\partial}{\partial n}\int_{\mathfrak{C}} \overset{0}{v}_n \ln\frac{1}{r}\,ds = 0, && \int_{\mathfrak{C}} \overset{0}{v}_n\,ds = 1,\\
&\overset{1}{v}_n + \frac{1}{\pi}\frac{\partial}{\partial n}\int_{\mathfrak{C}} \overset{1}{v}_n \ln\frac{1}{r}\,ds = 2y', && \int_{\mathfrak{C}} \overset{1}{v}_n\,ds = 0,\\
&\overset{2}{v}_n + \frac{1}{\pi}\frac{\partial}{\partial n}\int_{\mathfrak{C}} \overset{2}{v}_n \ln\frac{1}{r}\,ds = -2x', && \int_{\mathfrak{C}} \overset{2}{v}_n\,ds = 0,\\
&\overset{3}{v}_n + \frac{1}{\pi}\frac{\partial}{\partial n}\int_{\mathfrak{C}} \overset{3}{v}_n \ln\frac{1}{r}\,ds = k_t, && \int_{\mathfrak{C}} \overset{3}{v}_n\,ds = 0;
\end{aligned} \tag{53}$$

entsprechend lauten im Falle einer vorgegebenen Normalkomponente alle hiermit verträglichen Tangentialkomponenten

$$v_t = \Gamma\,\overset{0}{v}_t + v_{\infty,x}\overset{1}{v}_t + v_{\infty,y}\overset{2}{v}_t + \overset{3}{v}_t \tag{54}$$

mit den Definitionsgleichungen für die Grundlösungen

$$\begin{aligned}
&\overset{0}{v}_t + \frac{1}{\pi}\frac{\partial}{\partial n}\int_{\mathfrak{C}} \overset{0}{v}_t \ln\frac{1}{r}\,ds = 0, && \int_{\mathfrak{C}} \overset{0}{v}_t\,ds = 1,\\
&\overset{1}{v}_t + \frac{1}{\pi}\frac{\partial}{\partial n}\int_{\mathfrak{C}} \overset{1}{v}_t \ln\frac{1}{r}\,ds = 2x', && \int_{\mathfrak{C}} \overset{1}{v}_t\,ds = 0,\\
&\overset{2}{v}_t + \frac{1}{\pi}\frac{\partial}{\partial n}\int_{\mathfrak{C}} \overset{2}{v}_t \ln\frac{1}{r}\,ds = 2y', && \int_{\mathfrak{C}} \overset{2}{v}_t\,ds = 0,\\
&\overset{3}{v}_t + \frac{1}{\pi}\frac{\partial}{\partial n}\int_{\mathfrak{C}} \overset{3}{v}_t \ln\frac{1}{r}\,ds = -k_n, && \int_{\mathfrak{C}} \overset{3}{v}_t\,ds = 0.
\end{aligned} \tag{55}$$

Daß die willkürlichen Konstanten, deren Vorgabe das Problem jeweils eindeutig macht, in den allgemeinen Lösungen (52) und (54) die Bedeutung der Ergiebigkeit E und der Zirkulation Γ erhalten, folgt durch Integration dieser Gleichungen über $\mathfrak{C}$ unter Beachtung von (4) und (5) sowie der Nebenbedingungen, denen die Grundlösungen gemäß (53) und (55) unterliegen[1].

Die Grundlösungen lassen sich folgendermaßen anschaulich beschreiben:

$\overset{0}{v}_n$, $\overset{0}{v}_t$: „reines Ergiebigkeitsfeld", das $\mathfrak{C}$ als Potentiallinie, bzw. „reines Zirkulationsfeld", das $\mathfrak{C}$ als Feldlinie enthält, im Unendlichen verschwindend;

$\overset{1}{v}_n$, $\overset{1}{v}_t$: ergiebigkeitsfreies Feld mit $\mathfrak{C}$ als Potentiallinie bzw. zirkulationsfreies Feld mit $\mathfrak{C}$ als Feldlinie, beide im Unendlichen gegen (1, 0) gehend;

$\overset{2}{v}_n$, $\overset{2}{v}_t$: dto., nur im Unendlichen gegen (0, 1) gehend;

$\overset{3}{v}_n$, $\overset{3}{v}_t$: ergiebigkeitsfreies Feld mit vorgeschriebener Tangentialkomponente bzw. zirkulationsfreies Feld mit vorgeschriebener Normalkomponente, beide im Unendlichen verschwindend.

§ 6. Transformation auf einen beliebigen Kurvenparameter

Wir denken uns $\mathfrak{C}$ durch eine Parameterdarstellung $x(\varphi)$, $y(\varphi)$ mit den in § 2 genannten Eigenschaften gegeben. Nach Einführung neuer Funktionen ω_n, ω_t, $\varkappa_n$, $\varkappa_t$, $\overset{\alpha}{\omega}_n$, $\overset{\alpha}{\omega}_t$, $\alpha = 0 \ldots 3$, durch

$$v_n = \frac{\omega_n}{\underset{+}{\sqrt{\dot{x}^2+\dot{y}^2}}}, \qquad v_t = \frac{\omega_t}{\underset{+}{\sqrt{\dot{x}^2+\dot{y}^2}}}, \tag{56}$$

$$k_n = \frac{\varkappa_n}{\underset{+}{\sqrt{\dot{x}^2+\dot{y}^2}}}, \qquad k_t = \frac{\varkappa_t}{\underset{+}{\sqrt{\dot{x}^2+\dot{y}^2}}}, \tag{57}$$

$$\overset{\alpha}{v}_n = \frac{\overset{\alpha}{\omega}_n}{\underset{+}{\sqrt{\dot{x}^2+\dot{y}^2}}}, \qquad \overset{\alpha}{v}_t = \frac{\overset{\alpha}{\omega}_t}{\underset{+}{\sqrt{\dot{x}^2+\dot{y}^2}}}, \qquad \alpha = 0 \ldots 3, \tag{58}$$

transformieren sich (53) und (55) in

$$\begin{aligned}
&\overset{0}{\omega}_n(\varphi) - \frac{1}{2\pi}\int_0^{2\pi} K(\varphi,\psi)\,\overset{0}{\omega}_n(\psi)\,d\psi = 0, && \int_0^{2\pi} \overset{0}{\omega}_n(\varphi)\,d\varphi = 1,\\
&\overset{1}{\omega}_n(\varphi) - \frac{1}{2\pi}\int_0^{2\pi} K(\varphi,\psi)\,\overset{1}{\omega}_n(\psi)\,d\psi = 2\dot{y}(\varphi), && \int_0^{2\pi} \overset{1}{\omega}_n(\varphi)\,d\varphi = 0,\\
&\overset{2}{\omega}_n(\varphi) - \frac{1}{2\pi}\int_0^{2\pi} K(\varphi,\psi)\,\overset{2}{\omega}_n(\psi)\,d\psi = -2\dot{x}(\varphi), && \int_0^{2\pi} \overset{2}{\omega}_n(\varphi)\,d\varphi = 0,\\
&\overset{3}{\omega}_n(\varphi) - \frac{1}{2\pi}\int_0^{2\pi} K(\varphi,\psi)\,\overset{3}{\omega}_n(\psi)\,d\psi = \varkappa_t(\varphi), && \int_0^{2\pi} \overset{3}{\omega}_n(\varphi)\,d\varphi = 0,
\end{aligned} \tag{59}$$

---

[1] Die Dimensionen der Grundlösungen sind nicht einheitlich; $\overset{0}{v}_n$, $\overset{0}{v}_t$ stellen eine reziproke Länge dar, $\overset{1}{v}_n$, $\overset{1}{v}_t$, $\overset{2}{v}_n$, $\overset{2}{v}_t$ sind dimensionslos und $\overset{3}{v}_n$, $\overset{3}{v}_t$ besitzen dieselbe Dimension wie $|\mathfrak{v}|$.

bzw.

$$
(60)\qquad
\begin{aligned}
&\overset{0}{\omega}_t(\varphi) - \frac{1}{2\pi}\int_0^{2\pi} K(\varphi,\psi)\,\overset{0}{\omega}_t(\psi)\,d\psi = 0, &&\int_0^{2\pi}\overset{0}{\omega}_t(\varphi)\,d\varphi = 1,\\
&\overset{1}{\omega}_t(\varphi) - \frac{1}{2\pi}\int_0^{2\pi} K(\varphi,\psi)\,\overset{1}{\omega}_t(\psi)\,d\psi = 2\dot{x}(\varphi), &&\int_0^{2\pi}\overset{1}{\omega}_t(\varphi)\,d\varphi = 0,\\
&\overset{2}{\omega}_t(\varphi) - \frac{1}{2\pi}\int_0^{2\pi} K(\varphi,\psi)\,\overset{2}{\omega}_t(\psi)\,d\psi = 2\dot{y}(\varphi), &&\int_0^{2\pi}\overset{2}{\omega}_t(\varphi)\,d\varphi = 0,\\
&\overset{3}{\omega}_t(\varphi) - \frac{1}{2\pi}\int_0^{2\pi} K(\varphi,\psi)\,\overset{3}{\omega}_t(\psi)\,d\psi = -\varkappa_n(\varphi), &&\int_0^{2\pi}\overset{3}{\omega}_t(\varphi)\,d\varphi = 0,
\end{aligned}
$$

mit dem gemeinsamen Kern

$$
(61)\qquad K(\varphi,\psi) = 2\,\frac{\dot{y}(\varphi)\,[x(\varphi)-x(\psi)] - \dot{x}(\varphi)\,[y(\varphi)-y(\psi)]}{[x(\varphi)-x(\psi)]^2 + [y(\varphi)-y(\psi)]^2}\,.
$$

Dieser Kern ist mit 2π periodisch und außerdem stetig; denn es existiert der Grenzwert

$$
(62)\qquad \lim_{\psi\to\varphi+2k\pi} K(\varphi,\psi) = \frac{\dot{x}(\varphi)\,\ddot{y}(\varphi) - \dot{y}(\varphi)\,\ddot{x}(\varphi)}{[\dot{x}(\varphi)]^2 + [\dot{y}(\varphi)]^2}, \qquad k = 0, \pm 1, \pm 2 \ldots .
$$

Die Konjugierten k_n, k_t transformieren sich gemäß (42), (56) und (57) in

$$
(63)\qquad \varkappa(\varphi) = -\frac{1}{2\pi}\int_0^{2\pi} L(\varphi,\psi)\,\omega(\psi)\,d\psi,
$$

wobei ω durch ω_n bzw. ω_t und $\varkappa$ durch $\varkappa_n$ bzw. $\varkappa_t$ ersetzt zu denken ist und

$$
(64)\qquad L(\varphi,\psi) = 2\,\frac{\dot{x}(\varphi)\,[x(\varphi)-x(\psi)] + \dot{y}(\varphi)\,[y(\varphi)-y(\psi)]}{[x(\varphi)-x(\psi)]^2 + [y(\varphi)-y(\psi)]^2}
$$

ein mit 2π periodischer, für $\psi = \varphi + 2k\pi$, $k = 0, \pm 1, \pm 2 \ldots$; singulärer Kern ist. Das als Cauchyscher Hauptwert zu verstehende Integral (63) läßt sich einfacher auswerten, wenn man von $L(\varphi,\psi)$ einen stetigen Anteil $H(\varphi,\psi)$ abspaltet:

$$
(65)\qquad L(\varphi,\psi) = -\operatorname{ctg}\frac{\psi-\varphi}{2} + H(\varphi,\psi);
$$

die Stetigkeit von $H(\varphi,\psi)$ folgt aus der Existenz des Grenzwertes

$$
(66)\qquad \lim_{\psi\to\varphi+2k\pi} H(\varphi,\psi) = \frac{\dot{x}(\varphi)\,\ddot{x}(\varphi) + \dot{y}(\varphi)\,\ddot{y}(\varphi)}{[\dot{x}(\varphi)]^2 + [\dot{y}(\varphi)]^2}, \qquad k = 0, \pm 1, \pm 2 \ldots .
$$

Elimination von $L(\varphi,\psi)$ aus (63) und (65) ergibt

$$
(67)\qquad \varkappa(\varphi) = \frac{1}{2\pi}\int_0^{2\pi} \omega(\psi)\,\operatorname{ctg}\frac{\psi-\varphi}{2}\,d\psi - \frac{1}{2\pi}\int_0^{2\pi} H(\varphi,\psi)\,\omega(\psi)\,d\psi;
$$

damit ist die Bestimmung der $\varkappa_n$, $\varkappa_t$ im wesentlichen zurückgeführt auf die bekannte Aufgabe, zu einer stetigen und mit 2π periodischen Funktion $f(\varphi)$ die zugehörige „konjugiert harmonische“ Funktion

$$
(68)\qquad g(\varphi) = \frac{1}{2\pi}\int_0^{2\pi} f(\psi)\,\operatorname{ctg}\frac{\psi-\varphi}{2}\,d\psi
$$

aufzufinden. Das „Kotangensintegral“ (68) beherrscht man sowohl in der Theorie (die Fourier-Entwicklung für $g(\varphi)$ erhält man aus der für $f(\varphi)$, indem man die Konstante fortläßt und $\cos\mu\varphi$ durch $-\sin\mu\varphi$ und $\sin\mu\varphi$ durch $\cos\mu\varphi$, $\mu=1, 2, 3\ldots$, ersetzt) als auch in numerischer Hinsicht, worauf in § 7 näher eingegangen werden soll.

Die stetigen Kerne $K(\varphi,\psi)$ und $H(\varphi,\psi)$ lassen sich auch folgendermaßen darstellen:

$$K(\varphi,\psi)=2\frac{\partial}{\partial\varphi}\operatorname{arc\,tg}\frac{y(\varphi)-y(\psi)}{x(\varphi)-x(\psi)}, \tag{69}$$

$$H(\varphi,\psi)=\frac{\partial}{\partial\varphi}\ln\frac{[x(\varphi)-x(\psi)]^2+[y(\varphi)-y(\psi)]^2}{\sin^2\frac{\varphi-\psi}{2}}, \tag{70}$$

woraus die für das anschließend beschriebene numerische Verfahren wichtigen Identitäten

$$\int_0^{2\pi}K(\varphi,\psi)\,d\varphi=2\pi, \tag{71}$$

$$\int_0^{2\pi}H(\varphi,\psi)\,d\varphi=0 \tag{72}$$

folgen.

Ist speziell $\mathfrak{C}$ ein Kreis $\bigl(x=x_0+a\cos(\varphi-\varphi_0),\ y=y_0+a\sin(\varphi-\varphi_0)\bigr)$, so wird $K(\varphi,\psi)=1$ und $H(\varphi,\psi)=0$. Dann liefern (56), (59), (60) und (67) die bekannte Tatsache, daß $\overset{3}{v}_n(\varphi)$ gleich der harmonisch Konjugierten zu $v_t(\varphi)$ und $\overset{3}{v}_t(\varphi)$ gleich der harmonisch Konjugierten zu $-v_n(\varphi)$ ist.

§ 7. Diskrete Approximation der Grundlösungen

Wir gehen von einer Parameterdarstellung $x(\varphi)$, $y(\varphi)$ für $\mathfrak{C}$ mit den in § 2 genannten Eigenschaften aus. Es genügt jedoch, daß die Funktionen $x(\varphi)$, $y(\varphi)$ an $2N$ diskreten Stellen auf $\mathfrak{C}$ bekannt sind, die den äquidistanten Parameterwerten

$$\varphi_\mu=\frac{\mu\pi}{N}, \qquad \mu=0\ldots(2N-1), \tag{73}$$

entsprechen ($N\geqq 1$). Ist eine solche Parameterdarstellung nicht explizit bekannt, sondern $\mathfrak{C}$ z. B. graphisch gegeben, so kommt es offenbar nur darauf an, die Lage dieser „Stützstellen“ geeignet zu wählen. Es sei betont, daß die Stützstellen beliebig und somit ganz den praktischen Erfordernissen der numerischen Rechnung entsprechend auf $\mathfrak{C}$ verteilt werden können; die Verteilung braucht nur von einer gewissen Ebenmäßigkeit[1] zu sein, damit die anschließend beschriebenen numerischen Operationen (z. B. die Differentiation) sinnvoll bleiben (Fig. 1). Für bestimmte, praktisch interessante Profiltypen kann man derart geeignete Profilkoordinaten auf einfache Weise unmittelbar geometrisch konstruieren [7].

---

[1] Als Kriterium hierfür kann z. B. dienen, daß das durch trigonometrische Interpolation der Profilkoordinaten $x(\varphi_\mu)$, $y(\varphi_\mu)$, $\mu=0\ldots(2N-1)$, entstehende Profil von dem gegebenen (etwa im Rahmen der Zeichengenauigkeit) nicht unterschieden werden kann.

Zunächst ist zu einer an den Stellen (73) gegebenen und mit 2π periodischen Funktion $f(\varphi)$ die durch (68) erklärte harmonisch Konjugierte $g(\varphi)$ und die Ableitung $\dot{f}(\varphi)$ näherungsweise numerisch zu bestimmen[1]. Mit Hilfe der Methode der trigonometrischen Interpolation fand Garrick [8]

$$\tilde{g}_\mu = \sum_{\nu=1}^{2N-1} C_\nu f_{\mu+\nu}, \qquad \mu = 0 \ldots (2N-1)\,[2], \tag{74}$$

mit

$$C_\nu = \frac{1-(-1)^\nu}{2N} \operatorname{ctg} \frac{\varphi_\nu}{2}, \qquad \nu = 1 \ldots (2N-1). \tag{75}$$

Entsprechend ergibt sich für die Ableitung [7]

$$\tilde{\dot{f}}_\mu = \sum_{\nu=1}^{2N-1} D_\nu f_{\mu+\nu}, \qquad \mu = 0 \ldots (2N-1), \tag{76}$$

mit

$$D_\nu = -\frac{(-1)^\nu}{2} \operatorname{ctg} \frac{\varphi_\nu}{2}, \qquad \nu = 1 \ldots (2N-1). \tag{77}$$

Die Symmetrieeigenschaften der C_ν, D_ν bewirken

$$\sum_{\mu=0}^{2N-1} \tilde{g}_\mu = 0, \tag{78}$$

$$\sum_{\mu=0}^{2N-1} \tilde{\dot{f}}_\mu = 0. \tag{79}$$

Zur Berechnung der $\varkappa_n$, $\varkappa_t$ formen wir (67) zunächst mit Hilfe von (72) um in

$$\varkappa(\varphi) = \frac{1}{2\pi} \int_0^{2\pi} \omega(\psi) \operatorname{ctg} \frac{\psi-\varphi}{2}\, d\psi + \frac{1}{2\pi} \int_0^{2\pi} [H(\psi,\varphi)\,\omega(\varphi) - H(\varphi,\psi)\,\omega(\psi)]\, d\psi;$$

wertet man die auftretenden Integrale durch (74) bzw. durch die Rechteckformel[3] aus, so ergibt sich

$$\tilde{\varkappa}_\mu = \sum_{\nu=1}^{2N-1} C_\nu\, \omega_{\mu+\nu} + \frac{1}{2N} \sum_{\substack{\nu=0 \\ \nu \neq \mu}}^{2N-1} (\tilde{H}_{\nu\mu}\, \omega_\mu - \tilde{H}_{\mu\nu}\, \omega_\nu), \qquad \mu = 0 \ldots (2N-1), \tag{80}$$

wodurch zweierlei erreicht wird: erstens werden die Diagonalelemente (66) des Kernes $H(\varphi, \psi)$, deren numerische Berechnung wegen der vorkommenden zweiten Ableitungen mit erheblichen Fehlerquellen verbunden wäre, nicht benötigt, und zweitens gilt wegen (74) und (78)

$$\sum_{\mu=0}^{2N-1} \tilde{\varkappa}_\mu = 0. \tag{81}$$

---

[1] Ist $f(\varphi)$ bzw. $f(\varphi, \psi)$ in den vorkommenden Veränderlichen mit 2π periodisch, so wird im folgenden $f(\varphi_\mu) = f_\mu$ bzw. $f(\varphi_\mu, \varphi_\nu) = f_{\mu\nu}$ gesetzt. Näherungen zu $f(\varphi)$ bzw. $f(\varphi, \psi)$ werden mit $\tilde{f}(\varphi)$ bzw. $\tilde{f}(\varphi, \psi)$ bezeichnet.

[2] Für $\mu \geqq 2N$ ist f_μ durch $f_\mu = f_{\mu-2N}$ erklärt.

[3] Für diese im Falle periodischer Integranden bekanntlich sehr genaue Quadraturformel wurde kürzlich von Hämmerlin [9] eine Fehlerabschätzung angegeben.

Mit Hilfe von (65), (73) und (75) leiten wir einen Ausdruck für (80) her, in dem nur noch die Werte des (einfacher zu berechnenden) Kernes $L(\varphi, \psi)$ vorkommen. Es wird

$$\begin{aligned}\tilde{\varkappa}_\mu &= \sum_{\substack{\nu=0\\ \nu\neq\mu}}^{2N-1} \frac{1-(-1)^{\nu-\mu}}{2N}\,\omega_\nu \operatorname{ctg}\frac{(\nu-\mu)\pi}{2N} + \frac{1}{2N}\sum_{\substack{\nu=0\\ \nu\neq\mu}}^{2N-1} (\widetilde{L}_{\nu\mu}\,\omega_\mu - \widetilde{L}_{\mu\nu}\,\omega_\nu) + \\ &\quad + \frac{1}{2N}\sum_{\substack{\nu=0\\ \nu\neq\mu}}^{2N-1}\left(\omega_\mu \operatorname{ctg}\frac{(\mu-\nu)\pi}{2N} - \omega_\nu \operatorname{ctg}\frac{(\nu-\mu)\pi}{2N}\right) \\ &= -\frac{1}{2N}\sum_{\substack{\nu=0\\ \nu\neq\mu}}^{2N-1} (-1)^{\nu-\mu}\,\omega_\nu \operatorname{ctg}\frac{(\nu-\mu)\pi}{2N} + \frac{1}{2N}\sum_{\substack{\nu=0\\ \nu\neq\mu}}^{2N-1} (\widetilde{L}_{\nu\mu}\,\omega_\mu - \widetilde{L}_{\mu\nu}\,\omega_\nu), \\ &\qquad \mu = 0 \ldots (2N-1);\end{aligned}$$

hieraus folgt mit Hilfe von (76) und (77)

$$\tilde{\varkappa}_\mu = \frac{\dot{\tilde{\omega}}_\mu}{N} + \frac{1}{2N}\sum_{\substack{\nu=0\\ \nu\neq\mu}}^{2N-1} (\widetilde{L}_{\nu\mu}\,\omega_\mu - \widetilde{L}_{\mu\nu}\,\omega_\nu), \qquad \mu = 0 \ldots (2N-1). \tag{82}$$

Hierbei ist zu beachten, daß $\dot{\tilde{\omega}}_\mu$ nur formale Bedeutung zukommt; da nämlich $\omega(\varphi)$ nur als stetig vorausgesetzt ist, braucht die Ableitung $\dot{\omega}(\varphi)$ gar nicht zu existieren. Der praktische Vorteil, die Bezeichnung $\dot{\tilde{\omega}}_\mu$ dennoch zu verwenden, besteht darin, daß man zu den bekannten Daten x_μ, y_μ, ω_μ nur noch die Ableitung (76) bestimmen muß, die Formel (74) für die Berechnung der konjugiert harmonischen Funktion jedoch nicht mehr benötigt. Die ω_μ bzw. die $\omega_{n\mu}$ oder $\omega_{t\mu}$ sind bei gegebenem $v_{n\mu}$ oder $v_{t\mu}$ selbst Näherungen und aus (56) zu berechnen, wo man $\dot{x}_\mu$ und $\dot{y}_\mu$ durch $\dot{\tilde{x}}_\mu$ und $\dot{\tilde{y}}_\mu$ zu ersetzen hat.

Wir haben nunmehr Näherungen für die inhomogenen Seiten der Integralgleichungen (59) und (60) zur Verfügung. Diese Integralgleichungen führen nach einer einfachen Umformung mit (71) und nach anschließender Anwendung der Rechteckformel auf Systeme mit zunächst $2N+1$ linearen Gleichungen für $2N$ Unbekannte [*10*]; sie lauten im Falle einer vorgegebenen Tangentialkomponente

$$\begin{aligned} &\frac{1}{2N}\sum_{\substack{\nu=0\\ \nu\neq\mu}}^{2N-1} (\widetilde{K}_{\nu\mu}\,\overset{0}{\tilde{\omega}}_{n\mu} - \widetilde{K}_{\mu\nu}\,\overset{0}{\tilde{\omega}}_{n\nu}) = 0, &\qquad &\frac{1}{2N}\sum_{\mu=0}^{2N-1} \overset{0}{\tilde{\omega}}_{n\mu} = \frac{1}{2\pi}, \\ &\frac{1}{2N}\sum_{\substack{\nu=0\\ \nu\neq\mu}}^{2N-1} (\widetilde{K}_{\nu\mu}\,\overset{1}{\tilde{\omega}}_{n\mu} - \widetilde{K}_{\mu\nu}\,\overset{1}{\tilde{\omega}}_{n\nu}) = 2\dot{\tilde{y}}_\mu, &\qquad &\frac{1}{2N}\sum_{\mu=0}^{2N-1} \overset{1}{\tilde{\omega}}_{n\mu} = 0, \\ &\frac{1}{2N}\sum_{\substack{\nu=0\\ \nu\neq\mu}}^{2N-1} (\widetilde{K}_{\nu\mu}\,\overset{2}{\tilde{\omega}}_{n\mu} - \widetilde{K}_{\mu\nu}\,\overset{2}{\tilde{\omega}}_{n\nu}) = -2\dot{\tilde{x}}_\mu, &\qquad &\frac{1}{2N}\sum_{\mu=0}^{2N-1} \overset{2}{\tilde{\omega}}_{n\mu} = 0, \\ &\frac{1}{2N}\sum_{\substack{\nu=0\\ \nu\neq\mu}}^{2N-1} (\widetilde{K}_{\nu\mu}\,\overset{3}{\tilde{\omega}}_{n\mu} - \widetilde{K}_{\mu\nu}\,\overset{3}{\tilde{\omega}}_{n\nu}) = \tilde{\varkappa}_{t\mu}, &\qquad &\frac{1}{2N}\sum_{\mu=0}^{2N-1} \overset{3}{\tilde{\omega}}_{n\mu} = 0, \end{aligned} \tag{83}$$

und im Falle einer vorgegebenen Normalkomponente

$$
(84)\qquad
\begin{aligned}
&\frac{1}{2N}\sum_{\substack{\nu=0\\ \nu\neq\mu}}^{2N-1}\Big(\tilde{K}_{\nu\mu}\overset{0}{\tilde{\tilde{\omega}}}_{t\mu}-\tilde{K}_{\mu\nu}\overset{0}{\tilde{\tilde{\omega}}}_{t\nu}\Big)=0, &\qquad& \frac{1}{2N}\sum_{\mu=0}^{2N-1}\overset{0}{\tilde{\tilde{\omega}}}_{t\mu}=\frac{1}{2\pi},\\
&\frac{1}{2N}\sum_{\substack{\nu=0\\ \nu\neq\mu}}^{2N-1}\Big(\tilde{K}_{\nu\mu}\overset{1}{\tilde{\tilde{\omega}}}_{t\mu}-\tilde{K}_{\mu\nu}\overset{1}{\tilde{\tilde{\omega}}}_{t\nu}\Big)=2\dot{\tilde{x}}_{\mu}, && \frac{1}{2N}\sum_{\mu=0}^{2N-1}\overset{1}{\tilde{\tilde{\omega}}}_{t\mu}=0,\\
&\frac{1}{2N}\sum_{\substack{\nu=0\\ \nu\neq\mu}}^{2N-1}\Big(\tilde{K}_{\nu\mu}\overset{2}{\tilde{\tilde{\omega}}}_{t\mu}-\tilde{K}_{\mu\nu}\overset{2}{\tilde{\tilde{\omega}}}_{t\nu}\Big)=2\dot{\tilde{y}}_{\mu}, && \frac{1}{2N}\sum_{\mu=0}^{2N-1}\overset{2}{\tilde{\tilde{\omega}}}_{t\mu}=0,\\
&\frac{1}{2N}\sum_{\substack{\nu=0\\ \nu\neq\mu}}^{2N-1}\Big(\tilde{K}_{\nu\mu}\overset{3}{\tilde{\tilde{\omega}}}_{t\mu}-\tilde{K}_{\mu\nu}\overset{3}{\tilde{\tilde{\omega}}}_{t\nu}\Big)=-\tilde{\varkappa}_{n\mu}, && \frac{1}{2N}\sum_{\mu=0}^{2N-1}\overset{3}{\tilde{\tilde{\omega}}}_{t\mu}=0,
\end{aligned}
$$

wobei stets $\mu=0\ldots(2N-1)$. Die Systeme (83) und (84) enthalten sämtlich eine lineare Abhängigkeit und sind daher nur scheinbar überbestimmt: Die Summation über alle $\mu=0\ldots(2N-1)$ ergibt nämlich wegen (79) und (81) identisch Null. Infolgedessen kann man ohne Einschränkung z.B. jeweils die Gleichung für $\mu=0$ in (83) und (84) unterdrücken. Damit sind in jedem Fall vier Gleichungssysteme, bestehend aus $2N$ Gleichungen für $2N$ Unbekannte, zu lösen; diese Gleichungssysteme unterscheiden sich nicht durch ihre Koeffizientenmatrix, sondern nur durch ihre inhomogene Seite.

Die in (83) und (84) auftretenden Näherungswerte für die Kerne $K(\varphi,\psi)$ und $L(\varphi,\psi)$ ergeben sich aus (61) und (64) zu

$$
(85)\qquad \tilde{K}_{\mu\nu}=2\,\frac{\dot{\tilde{y}}_{\mu}(x_{\mu}-x_{\nu})-\dot{\tilde{x}}_{\mu}(y_{\mu}-y_{\nu})}{(x_{\mu}-x_{\nu})^2+(y_{\mu}-y_{\nu})^2},\qquad \mu,\nu=0\ldots(2N-1),\quad \mu\neq\nu,
$$

$$
(86)\qquad \tilde{L}_{\mu\nu}=2\,\frac{\dot{\tilde{x}}_{\mu}(x_{\mu}-x_{\nu})+\dot{\tilde{y}}_{\mu}(y_{\mu}-y_{\nu})}{(x_{\mu}-x_{\nu})^2+(y_{\mu}-y_{\nu})^2},\qquad \mu,\nu=0\ldots(2N-1),\quad \mu\neq\nu.
$$

Mit den Lösungen von (83) bzw. (84) beherrscht man schließlich durch (58) die gesuchten Näherungen für die Grundlösungen des Problems

$$
(87)\qquad \overset{\alpha}{\tilde{v}}_{n\mu}=\frac{\overset{\alpha}{\tilde{\tilde{\omega}}}_{n\mu}}{\underset{+}{\sqrt{\dot{\tilde{x}}_{\mu}^2+\dot{\tilde{y}}_{\mu}^2}}},\qquad \overset{\alpha}{\tilde{v}}_{t\mu}=\frac{\overset{\alpha}{\tilde{\tilde{\omega}}}_{t\mu}}{\underset{+}{\sqrt{\dot{\tilde{x}}_{\mu}^2+\dot{\tilde{y}}_{\mu}^2}}},\qquad \alpha=0\ldots3,\ \mu=0\ldots(2N-1).
$$

§ 8. Numerisches Beispiel

Zu einer vorgegebenen Konstanten $a>0$ sind $z^*=\pm a$ Fixpunkte der nach Joukowski benannten konformen Abbildung

$$
(88)\qquad z=\frac{1}{2}\left(z^*+\frac{a^2}{z^*}\right).
$$

Ein „Joukowski-Profil im engeren Sinne“ erhält man aus (88) bei Abbildung eines Kreises der z^*-Ebene, der den Punkt $z^*=a$ auf dem Rande und $z^*=-a$

im Inneren enthält, in die z-Ebene. Ein solches Profil besitzt jedoch bei $z=a$ eine „scharfe Hinterkante“, ist also nicht stetig gekrümmt und entspricht somit nicht den in § 2 genannten Voraussetzungen. Diese werden dagegen von einem „verallgemeinerten Joukowski-Profil“ erfüllt, das durch Abbildung eines Kreises entsteht, der beide Fixpunkte im Inneren enthält, z. B.:

$$(89)\qquad \begin{aligned} x^* &= a\left\{-0{,}2 + \left(0{,}1+\sqrt{1{,}6}\right)\frac{1{,}2\cos\varphi + 0{,}4\sin\varphi}{\sqrt{1{,}6}}\right\},\\ y^* &= a\left\{0{,}4 + \left(0{,}1+\sqrt{1{,}6}\right)\frac{1{,}2\sin\varphi - 0{,}4\cos\varphi}{\sqrt{1{,}6}}\right\}. \end{aligned}$$

Dieser Kreis mit dem Mittelpunkt $(-0{,}2a,\ 0{,}4a)$ und dem Radius $\left(0{,}1+\sqrt{1{,}6}\right)a$ wird mit wachsendem $0\leqq\varphi\leqq 2\pi$ einmal im Gegenuhrzeigersinn durchlaufen; der kürzeste Abstand zum Fixpunkt $(a, 0)$ wird für $\varphi=0$ angenommen und beträgt $0{,}1a$. In der z-Ebene bekommt man daher für $\varphi=0$ eine „abgerundete Hinterkante“.

Mit einer Konstanten $c>0$ laute die Normalkomponente auf dem Kreis (89)

$$(90)\qquad v_n^* = \frac{c}{2\pi}\left(1 - 0{,}7\cos\varphi - 0{,}1\cos 2\varphi - 1{,}8\sin\varphi + 0{,}4\sin 2\varphi\right),$$

so daß die Ergiebigkeit $E^*=\left(0{,}1+\sqrt{1{,}6}\right)ac$ beträgt. Da man die Verhältnisse beim Kreis geschlossen beherrscht[1], ergeben sich die Grundlösungen in der z^*-Ebene zu

$$(91)\qquad \begin{aligned} \overset{0}{v}{}_t^* &= \frac{1}{2\pi a\left(0{,}1+\sqrt{1{,}6}\right)},\\ \overset{1}{v}{}_t^* &= -2\,\frac{1{,}2\sin\varphi - 0{,}4\cos\varphi}{\sqrt{1{,}6}},\\ \overset{2}{v}{}_t^* &= 2\,\frac{1{,}2\cos\varphi + 0{,}4\sin\varphi}{\sqrt{1{,}6}},\\ \overset{3}{v}{}_t^* &= \frac{c}{2\pi}\left(-0{,}7\sin\varphi - 0{,}1\sin 2\varphi + 1{,}8\cos\varphi - 0{,}4\cos 2\varphi\right). \end{aligned}$$

Durch Einsetzen von (89) in (88) und anschließende Zerlegung in Real- und Imaginärteil bekommen wir die Parameterdarstellung des Joukowski-Profils $\mathfrak{C}$ (Fig. 1)

$$(92)\qquad \begin{aligned} x(\varphi) &= \frac{x^*}{2}\,\frac{x^{*2}+y^{*2}+a^2}{x^{*2}+y^{*2}},\\ y(\varphi) &= \frac{y^*}{2}\,\frac{x^{*2}+y^{*2}-a^2}{x^{*2}+y^{*2}}. \end{aligned}$$

Bei der Transformation der Grundlösungen in die z-Ebene ist zu beachten, daß für $\overset{0}{v}{}_t^*$ die zu 1 normierte Zirkulation, dagegen für $\overset{1}{v}{}_t^*$ und $\overset{2}{v}{}_t^*$ die Verhältnisse im Unendlichen erhalten bleiben müssen. Die Transformation von v_n^* und $\overset{3}{v}{}_t^*$ ist zunächst nur bis auf einen willkürlichen konstanten Faktor bestimmt; sie

---

[1] Vgl. § 1 und § 6.

Tabelle. *Koordinaten, Ergiebigkeitsverteilung und Grundlösungen*

μ	$\frac{x_\mu}{a}$	$\frac{y_\mu}{a}$	$\frac{v_{n\mu}}{c}$	$a\overset{0}{v}_{t\mu}$	$a\overset{0}{\tilde{v}}_{t\mu}$
0	1,00372944	−0,00263234	0,36455230	1,33544	1,33470
1	0,99706754	0,02659062	0,03287987	0,58889	0,58808
2	0,95853214	0,07731212	−0,04435321	0,34975	0,35004
3	0,89407013	0,14402183	−0,07573595	0,26447	0,26485
4	0,80795896	0,22041844	−0,09661890	0,22352	0,22326
5	0,70342879	0,30037568	−0,11454827	0,20099	0,20111
6	0,58321864	0,37838693	−0,13140922	0,18788	0,18784
7	0,44995579	0,44973820	−0,14695453	0,18030	0,18033
8	0,30637044	0,51056031	−0,15969286	0,17635	0,17634
9	0,15538841	0,55783464	−0,16723415	0,17503	0,17504
10	0,00014189	0,58938314	−0,16652219	0,17581	0,17581
11	−0,15607167	0,60385276	−0,15405499	0,17844	0,17844
12	−0,30986166	0,60069538	−0,12610049	0,18285	0,18285
13	−0,45783315	0,58014138	−0,07887795	0,18917	0,18917
14	−0,59668177	0,54316478	−0,00865403	0,19768	0,19768
15	−0,72328691	0,49143820	0,08831399	0,20885	0,20885
16	−0,83479631	0,42727697	0,21607123	0,22346	0,22346
17	−0,92869613	0,35357225	0,37957648	0,24266	0,24266
18	−1,00286000	0,27371329	0,58559860	0,26815	0,26815
19	−1,05557070	0,19149896	0,84315222	0,30210	0,30211
20	−1,08550654	0,11103651	1,15931231	0,34613	0,34613
21	−1,09168362	0,03662270	1,51383377	0,39574	0,39574
22	−1,07334466	−0,02740560	1,78595509	0,42624	0,42624
23	−1,02978710	−0,07687654	1,76264769	0,39857	0,39857
24	−0,96013491	−0,10809100	1,46203978	0,32408	0,32408
25	−0,86309224	−0,11830853	1,11191840	0,24954	0,24954
26	−0,73679688	−0,10650910	0,82600690	0,19366	0,19366
27	−0,57904531	−0,07449052	0,61497448	0,15537	0,15537
28	−0,38836315	−0,02813491	0,46414673	0,13038	0,13038
29	−0,16641745	0,02190366	0,35848934	0,11563	0,11563
30	0,07848031	0,06163036	0,28646980	0,10975	0,10975
31	0,32896981	0,07863289	0,23972911	0,11314	0,11314
32	0,56101920	0,06900911	0,21258381	0,12877	0,12878
33	0,75197631	0,04085216	0,20229516	0,16516	0,16517
34	0,88889492	0,01018559	0,21162528	0,24842	0,24844
35	0,97062747	−0,00740032	0,26355763	0,49206	0,49209

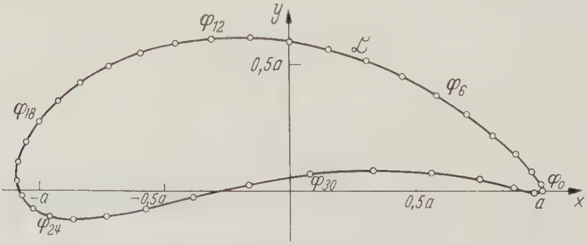

Fig. 1. Verallgemeinertes Joukowski-Profil mit $2N = 36$ Konturpunkten

läßt sich jedoch im Falle nicht verschwindender Ergiebigkeit durch die Forderung $E = E^*$ eindeutig festlegen. Die Normalkomponente (90) transformiert sich *für ein verallgemeinertes Joukowski-Profil* (2N=36)

$\overset{1}{v}_{t\mu}$	$\overset{1}{\tilde{v}}_{t\mu}$	$\overset{2}{v}_{t\mu}$	$\overset{2}{\tilde{v}}_{t\mu}$	$\frac{\overset{3}{v}_{t\mu}}{c}$	$\frac{\overset{3}{\tilde{v}}_{t\mu}}{c}$
3,62168	3,62595	10,86503	10,85866	2,55187	2,55444
0,74081	0,75476	4,99566	4,98408	0,99751	0,99873
−0,08192	−0,08140	2,99832	2,99975	0,51620	0,51476
−0,45471	−0,46279	2,22204	2,22745	0,33291	0,33247
−0,70458	−0,70066	1,78275	1,78013	0,23217	0,23289
−0,90231	−0,90367	1,46869	1,46955	0,16234	0,16192
−1,06902	−1,06862	1,20555	1,20536	0,10442	0,10463
−1,21122	−1,21149	0,96120	0,96131	0,04922	0,04911
−1,32995	−1,32982	0,72015	0,72014	−0,00845	−0,00840
−1,42406	−1,42417	0,47469	0,47471	−0,07167	−0,07170
−1,49145	−1,49142	0,22117	0,22118	−0,14202	−0,14202
−1,52969	−1,52972	−0,04179	−0,04178	−0,21986	−0,21986
−1,53630	−1,53632	−0,31438	−0,31437	−0,30439	−0,30439
−1.50878	−1,50880	−0,59630	−0,59629	−0,39384	−0,39384
−1,44445	−1,44446	−0,88742	−0,88740	−0,48561	−0,48561
−1,34010	−1,34011	−1,18833	−1,18832	−0,57646	−0,57646
−1.19127	−1,19128	−1,50112	−1,50111	−0,66277	−0,66277
−0,99092	−0,99094	−1,83001	−1,82999	−0,74055	−0,74055
−0,72721	−0,72723	−2,18163	−2,18162	−0,80520	−0,80520
−0,38004	−0,38006	−2,56282	−2,56280	−0,84992	−0,84992
0,08107	0,08105	−2,96726	−2,96724	−0,86111	−0,86111
0,68040	0,68037	−3,32495	−3,32492	−0,80776	−0,80777
1,34358	1,34355	−3,39956	−3,39953	−0,63814	−0,63814
1,78926	1,78923	−2,91238	−2,91235	−0,35350	−0,35350
1,84398	1,84395	−2,07947	−2,07945	−0,07979	−0,07979
1,67634	1,67631	−1,33031	−1,33029	0,09683	0,09683
1,46043	1,46041	−0,79080	−0,79079	0,18991	0,18991
1,26404	1,26402	−0,42135	−0,42133	0,23327	0,23326
1,10601	1,10599	−0,16401	−0,16399	0,25127	0,25127
0,99124	0,99121	0,02708	0,02710	0,25947	0,25947
0,92207	0,92204	0,18869	0,18871	0,26855	0,26855
0,90238	0,90234	0,35664	0,35668	0,28742	0,28742
0,94095	0,94087	0,57809	0,57816	0,32654	0,32653
1,05977	1,05960	0,93975	0,93991	0,40475	0,40473
1,32434	1,32384	1,66881	1,66924	0,57260	0,57258
2,00934	2,00766	3,71079	3,71196	1,04270	1,04287

somit in

$$v_n(\varphi) = v_n^* \left|\frac{dz^*}{dz}\right|_{\mathfrak{C}}, \tag{93}$$

während die Grundlösungen (91) in

$$\begin{aligned}
\overset{0}{v}_t(\varphi) &= \overset{0}{v}{}_t^* \left|\frac{dz^*}{dz}\right|_{\mathfrak{C}}, \\
\overset{1}{v}_t(\varphi) &= \overset{1}{v}{}_t^* \left|\frac{dz^*}{dz}\right|_{\mathfrak{C}} \left|\frac{dz}{dz^*}\right|_{\infty}, \\
\overset{2}{v}_t(\varphi) &= \overset{2}{v}{}_t^* \left|\frac{dz^*}{dz}\right|_{\mathfrak{C}} \left|\frac{dz}{dz^*}\right|_{\infty}, \\
\overset{3}{v}_t(\varphi) &= \overset{3}{v}{}_t^* \left|\frac{dz^*}{dz}\right|_{\mathfrak{C}}
\end{aligned} \tag{94}$$

übergehen; dabei ist wegen (88)

$$\left|\frac{dz^*}{dz}\right|_{\mathfrak{C}} = \left|\frac{1}{2}\left(1-\frac{a^2}{z^{*2}}\right)\right|^{-1} = \frac{2(x^{*2}+y^{*2})}{\sqrt[+]{(x^{*2}+y^{*2}+a^2)^2-4a^2x^{*2}}}, \tag{95}$$

$$\left|\frac{dz}{dz^*}\right|_{\infty} = \frac{1}{2} \tag{96}$$

einzusetzen.

Dieses Beispiel wurde für $2N=36$ Konturpunkte mit der elektronischen Rechenmaschine $G\,2$ (mittlere Rechengeschwindigkeit: 20 Operationen pro Sekunde) des Max-Planck-Instituts für Physik und Astrophysik in München nach dem in § 7 beschriebenen Verfahren numerisch berechnet. Die Ergebnisse (Tabelle) zeigen im Vergleich mit der strengen Lösung (94) eine Genauigkeit bis zu fünf Dezimalstellen; in einer Umgebung von $\varphi=0$, wo ungenauere Werte wegen der starken Krümmung von vornherein zu erwarten waren, liefert das Verfahren immerhin noch eine den praktischen Erfordernissen genügende Genauigkeit. Die $G\,2$ benötigte für dieses Beispiel eine Rechenzeit von 3 Std 25 min; daran ist die Auflösung des linearen Gleichungssystems (36 Unbekannte, 4 inhomogene Seiten), die nach dem Eliminationsverfahren von Gauss-Jordan durchgeführt wurde, mit 1 Std 45 min beteiligt.

§ 9. Erweiterung auf Gitterprofile

In der Strömungslehre interessieren die inkompressiblen Potentialströmungen auf und außerhalb einer gitterförmigen Anordnung von Profilen [7]. Ein derartiges „Gitterprofil" bestehe aus unendlich vielen kongruenten, einfach zusammenhängenden Gebieten $\mathfrak{B}_k$, $k=0, \pm 1, \pm 2 \ldots$, mit doppelpunktfreien, stetig gekrümmten Berandungen $\mathfrak{C}_k$, $k=0, \pm 1, \pm 2 \ldots$, die mit gleichem Abstand $t>0$ („Gitterteilung") in y-Richtung angeordnet sind. Die $\mathfrak{C}_k$, $k=0, \pm 1, \pm 2 \ldots$, sollen sich gegenseitig weder berühren noch überschneiden. Die Menge der außerhalb des Gitterprofils gelegenen Punkte sei $\mathfrak{A}$ genannt (Fig. 2). Für $\mathfrak{C}_0$ existiere eine Parameterdarstellung $x(\varphi)$, $y(\varphi)$ mit den in § 2 genannten Eigenschaften. Dann ist eine beliebige Berandung $\mathfrak{C}_k$ gegeben durch $x(\varphi)$, $y(\varphi)+kt$, $k=0, \pm 1, \pm 2 \ldots$. Die Konstanten

$$a_+ = \operatorname{Max}\{x(\varphi)\}, \qquad a_- = \operatorname{Min}\{x(\varphi)\} \tag{97}$$

legen die seitliche Begrenzung des Gitterprofils fest.

In $\mathfrak{A}$ und auf den $\mathfrak{C}_k$, $k=0, \pm 1, \pm 2 \ldots$, sei ein Klasse von Vektorfeldern $\mathfrak{v}(x, y)$ mit folgenden Eigenschaften Gegenstand der Betrachtung:

$\mathfrak{v}$ ist in $\mathfrak{A}$ und auf den $\mathfrak{C}_k$, $k=0, \pm 1, \pm 2 \ldots$, stetig;

$\mathfrak{v}$ ist in $\mathfrak{A}$ stetig differenzierbar und harmonisch;

$\mathfrak{v}$ ist in $\mathfrak{A}$ und auf den $\mathfrak{C}_k$, $k=0, \pm 1, \pm 2 \ldots$, hinsichtlich y periodisch mit t:

$$\mathfrak{v}(x, y+t) = \mathfrak{v}(x, y); \tag{98}$$

es existieren die Grenzwerte

$$\lim_{x\to\pm\infty}\left|e^{\mp\frac{2\pi x}{t}}\,\mathfrak{v}(x,\eta)\right| = 0 \tag{99}$$

für alle η eines beliebigen Intervalles $y-\frac{t}{2}\leqq\eta\leqq y+\frac{t}{2}$ im Sinne gleichmäßiger Konvergenz.

Ergiebigkeit und Zirkulation werden analog zu (4) und (5) definiert, wobei nur jeweils über eine beliebige Berandung, z.B. über $\mathfrak{C}_0$, integriert wird.

Satz 3[1]. *Der über irgendeinen Periodenstreifen genommene Mittelwert von* $\mathfrak{v}$ *ist rechts bzw. links vom Gitter gleich einer vektoriellen Konstanten*

$$\mathfrak{v}_{+\infty}=(v_{+\infty,x},v_{+\infty,y})\quad\text{bzw.}\quad\mathfrak{v}_{-\infty}=(v_{-\infty,x},v_{-\infty,y}),$$

d.h. es gilt für alle $a'_{+}>a_{+}$ *und* $a'_{-}<a_{-}$ *und alle* y

$$\frac{1}{t}\int\limits_{y-\frac{t}{2}}^{y+\frac{t}{2}}\mathfrak{v}(a'_{\pm},\eta)\,d\eta=\mathfrak{v}_{\pm\infty}. \tag{100}$$

Beweis. Betrachten wir ein rechts vom Gitter gelegenes einfach zusammenhängendes Gebiet $\mathfrak{B}$ mit doppelpunktfreier, stückweise glatter Berandung $\mathfrak{S}$. Oberer und unterer Teil von $\mathfrak{S}$ sollen kongruent sein und in entsprechenden Punkten mit gleicher Abszisse den Abstand t besitzen. Rechter und linker Teil von $\mathfrak{S}$ seien gegeben durch die Geradenstücke

$$\xi=x_1,\quad y_1-\frac{t}{2}\leqq\eta\leqq y_1+\frac{t}{2}\quad\text{bzw.}\quad\xi=x_0,\quad y_0-\frac{t}{2}\leqq\eta\leqq y_0+\frac{t}{2},$$

wobei (x_0,y_0) und (x_1,y_1) zwei Punkte mit $x_1>x_0>a_{+}$ sind. Auf Grund des Gaußschen und Stokesschen Integralsatzes gilt dann:

$$\frac{1}{t}\iint\limits_{\mathfrak{B}}\operatorname{div}\mathfrak{v}\,dx\,dy=\frac{1}{t}\int\limits_{\mathfrak{S}}v_n\,ds=\frac{1}{t}\int\limits_{y_1-\frac{t}{2}}^{y_1+\frac{t}{2}}v_x(x_1,\eta)\,d\eta-\frac{1}{t}\int\limits_{y_0-\frac{t}{2}}^{y_0+\frac{t}{2}}v_x(x_0,\eta)\,d\eta=0,$$

$$\frac{1}{t}\iint\limits_{\mathfrak{B}}\operatorname{rot}_z\mathfrak{v}\,dx\,dy=\frac{1}{t}\int\limits_{\mathfrak{S}}v_t\,ds=\frac{1}{t}\int\limits_{y_1-\frac{t}{2}}^{y_1+\frac{t}{2}}v_y(x_1,\eta)\,d\eta-\frac{1}{t}\int\limits_{y_0-\frac{t}{2}}^{y_0+\frac{t}{2}}v_y(x_0,\eta)\,d\eta=0.$$

Vektorielle Zusammenfassung ergibt die Existenz eines $\mathfrak{v}_{-\infty}$ mit der Eigenschaft (100). Der Beweis für die Existenz eines vektoriellen Mittelwertes $\mathfrak{v}_{-\infty}$ links vom Gitter verläuft analog.

Satz 4. *Die Differenz der Mittelwerte* (100) *ist durch die Ergiebigkeit und die Zirkulation für das Profil* $\mathfrak{C}_0$ *bestimmt:*

$$\mathfrak{v}_{+\infty}-\mathfrak{v}_{-\infty}=\frac{E}{t}\,\mathfrak{i}+\frac{\Gamma}{t}\,\mathfrak{j}. \tag{101}$$

---

[1] Dieser Satz wird in [7] unter der (weniger allgemeinen) Voraussetzung der Existenz einer einwertigen Stromfunktion bewiesen.

Beweis. Durch Anwendung des Gaußschen und Stokesschen Satzes auf das zweifach zusammenhängende Gebiet $\mathfrak{G}_k$ mit den Berandungen $\mathfrak{S}_k$ und $\mathfrak{C}_k$ (Fig. 2) erhält man

$$\frac{1}{t}\iint\limits_{\mathfrak{G}_k} \operatorname{div} \mathfrak{v}\, d x\, d y = \frac{1}{t} \int\limits_{y_+ - \frac{t}{2}}^{y_+ + \frac{t}{2}} v_x(a'_+, \eta)\, d\eta - \frac{1}{t} \int\limits_{y_- - \frac{t}{2}}^{y_- + \frac{t}{2}} v_x(a'_-, \eta)\, d\eta - \frac{1}{t} \int\limits_{\mathfrak{C}_k} v_n\, d s = 0,$$

$$\frac{1}{t}\iint\limits_{\mathfrak{G}_k} \operatorname{rot}_z \mathfrak{v}\, d x\, d y = \frac{1}{t} \int\limits_{v_+ - \frac{t}{2}}^{y_+ + \frac{t}{2}} v_y(a'_+, \eta)\, d\eta - \frac{1}{t} \int\limits_{y_- - \frac{t}{2}}^{y_- + \frac{t}{2}} v_y(a'_-, \eta)\, d\eta - \frac{1}{t} \int\limits_{\mathfrak{C}_k} v_t\, d s = 0.$$

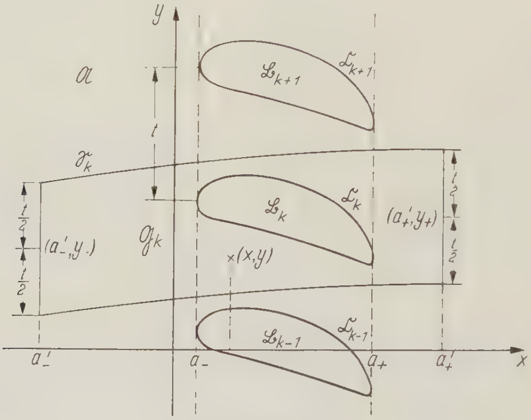

Fig. 2. Gitterprofil

Diese Gleichungen sind wegen (4), (5), (98) und (100) identisch mit der Komponentenzerlegung von (101).

Satz 5. *In* $\mathfrak{A}$ *wird* $\mathfrak{v}$ *dargestellt durch*

$$\mathfrak{v} = \mathfrak{v}_\infty - \operatorname{grad} \Phi + \operatorname{rot}(\Psi \mathfrak{k}), \tag{102}$$

wobei $\mathfrak{v}_\infty = (v_{\infty, x}, v_{\infty, y})$ *das vektorielle Mittel*

$$\mathfrak{v}_\infty = \frac{1}{2}(\mathfrak{v}_{+\infty} + \mathfrak{v}_{-\infty}) \tag{103}$$

ist und

$$\Phi = \frac{1}{2\pi} \int\limits_{\mathfrak{C}_0} v_n F\, d s, \tag{104}$$

$$\Psi = \frac{1}{2\pi} \int\limits_{\mathfrak{C}_0} v_t F\, d s \tag{105}$$

harmonische und hinsichtlich y periodische Funktionen sind mit

$$F(x, y; \xi, \eta) = \ln \frac{1}{\sqrt{2\left[\operatorname{Cof} \frac{2\pi(x-\xi)}{t} - \cos \frac{2\pi(y-\eta)}{t}\right]}}. \tag{106}$$

Beweis. Ein beliebiger in $\mathfrak{A}$ gelegener Aufpunkt (x, y) läßt sich immer als innerer Punkt eines zweifach zusammenhängenden Gebietes $\mathfrak{G}_k$ mit den Berandungen $\mathfrak{S}_k$ und $\mathfrak{C}_k$ auffassen (Fig. 2). Dort erfüllt (106) die Voraussetzungen zum Lemma aus § 3; insbesondere ist $F' = F - \ln \frac{1}{r}$ in einer Umgebung von (x, y) analytisch [7]. Es gilt daher

$$\mathfrak{v} = -\operatorname{grad}(\Phi^* + \Phi) + \operatorname{rot}\big((\Psi^* + \Psi)\,\mathfrak{k}\big) \tag{107}$$

mit

$$\Phi^* = -\frac{1}{2\pi}\int\limits_{\mathfrak{S}_k} v_n F\,ds, \qquad \Phi = \frac{1}{2\pi}\int\limits_{\mathfrak{C}_k} v_n F\,ds,$$

$$\Psi^* = -\frac{1}{2\pi}\int\limits_{\mathfrak{S}_k} v_t F\,ds, \qquad \Psi = \frac{1}{2\pi}\int\limits_{\mathfrak{C}_k} v_t F\,ds.$$

Wir untersuchen den Ausdruck

$$\mathfrak{w} = -\operatorname{grad}\Phi^* + [\operatorname{grad}\Psi^*, \mathfrak{k}]. \tag{108}$$

Bei der Auswertung der Integrale auf $\mathfrak{S}_k$ liefern die Anteile des oberen und unteren Wegabschnittes infolge der Periodizitätseigenschaften von $\mathfrak{v}$ und F zusammen keinen Beitrag. Wir können daher (108) gemäß $\mathfrak{w} = \mathfrak{w}_+ + \mathfrak{w}_-$ in zwei Anteile zerlegen, die den Integrationen über die beiden Geradenstücke rechts und links vom Gitter entsprechen. Zieht man die Gradienten jeweils unter die Integralzeichen, so findet man durch elementare Rechnung

$$\mathfrak{w}_+ = -\frac{1}{2t}\int\limits_{y_+ - \frac{t}{2}}^{y_+ + \frac{t}{2}} \left\{ \mathfrak{v}(a'_+, \eta)\,\frac{\mathfrak{Sin}\,\dfrac{2\pi(x - a'_+)}{t}}{\mathfrak{Cof}\,\dfrac{2\pi(x - a'_+)}{t} - \cos\dfrac{2\pi(y-\eta)}{t}} + \right.$$
$$\left. + [\mathfrak{k}, \mathfrak{v}(a'_+, \eta)]\,\frac{\sin\dfrac{2\pi(y-\eta)}{t}}{\mathfrak{Cof}\,\dfrac{2\pi(x - a'_+)}{t} - \cos\dfrac{2\pi(y-\eta)}{t}} \right\} d\eta.$$

Bei Berücksichtigung von (100) folgt

$$\mathfrak{w}_+ = \frac{1}{2}\mathfrak{v}_{+\infty} - \frac{1}{t}\int\limits_{y_+ - \frac{t}{2}}^{y_+ + \frac{t}{2}} \left\{ e^{-\frac{2\pi a'_+}{t}}\,\mathfrak{v}(a'_+, \eta)\,P(x, y; a'_+, \eta) + \right. \tag{109}$$
$$\left. + \left[\mathfrak{k}, e^{-\frac{2\pi a'_+}{t}}\,\mathfrak{v}(a'_+, \eta)\right] Q(x, y; a'_+, \eta) \right\} d\eta$$

mit

$$P(x, y; a'_+, \eta) = e^{\frac{2\pi x}{t}}\,\frac{e^{\frac{2\pi(x - a'_+)}{t}} - \cos\dfrac{2\pi(y-\eta)}{t}}{1 - 2e^{\frac{2\pi(x - a'_+)}{t}}\cos\dfrac{2\pi(y-\eta)}{t} + e^{\frac{4\pi(x - a'_+)}{t}}}, \tag{110}$$

$$Q(x, y; a'_+, \eta) = e^{\frac{2\pi x}{t}}\,\frac{\sin\dfrac{2\pi(y-\eta)}{t}}{1 - 2e^{\frac{2\pi(x - a'_+)}{t}}\cos\dfrac{2\pi(y-\eta)}{t} + e^{\frac{4\pi(x - a'_+)}{t}}}. \tag{111}$$

Diese Ausdrücke sind wegen $a'_+ > x$ für alle $a''_+ \geqq a'_+$ und alle $y_+ - \frac{t}{2} \leqq \eta \leqq y_+ + \frac{t}{2}$ gleichmäßig beschränkt:

$$|P(x, y; a''_+, \eta)| \leqq e^{\frac{2\pi x}{t}} \frac{1 + e^{\frac{2\pi(x - a'_+)}{t}}}{\left(1 - e^{\frac{2\pi(x - a'_+)}{t}}\right)^2},$$

$$|Q(x, y; a''_+, \eta)| \leqq e^{\frac{2\pi x}{t}} \frac{1}{\left(1 - e^{\frac{2\pi(x - a'_+)}{t}}\right)^2}.$$

Im Zusammenhang mit der Voraussetzung (99) konvergiert der Integrand in (109) für alle $y_+ - \frac{t}{2} \leqq \eta \leqq y_+ + \frac{t}{2}$ gleichmäßig gegen Null, wenn $a'_+ \to +\infty$ strebt. Folglich existieren die Grenzwerte

$$\lim_{a'_+ \to +\infty} \mathfrak{w}_+ = \tfrac{1}{2} \mathfrak{v}_{+\infty}.$$

Nach einer entsprechenden Überlegung für $\mathfrak{w}_-$ ergibt sich

$$\lim_{\substack{a'_+ \to +\infty \\ a'_- \to -\infty}} \mathfrak{w} = \lim_{\substack{a'_+ \to +\infty \\ a'_- \to -\infty}} (\mathfrak{w}_+ + \mathfrak{w}_-) = \tfrac{1}{2} (\mathfrak{v}_{+\infty} + \mathfrak{v}_{-\infty}) = \mathfrak{v}_\infty.$$

Zusammen mit (12), (107) und (108) folgt die Behauptung (102) bis (106) bis auf den geringfügigen Unterschied, daß die Integrale für Φ und Ψ zunächst noch über $\mathfrak{C}_k$ zu erstrecken sind. Auf Grund der Periodizitätseigenschaften ändern diese Integrale ihre Werte jedoch nicht, wenn sie über $\mathfrak{C}_0$ statt über $\mathfrak{C}_k$ erstreckt werden.

Satz 6. *Im Unendlichen konvergiert* $\mathfrak{v}$ *gleichmäßig gegen die konstanten Mittelwerte im Endlichen* $\mathfrak{v}_{\pm\infty}$, *d.h. für jedes Intervall* $y - \frac{t}{2} \leqq \eta \leqq y + \frac{t}{2}$ *existieren die Grenzwerte*

$$\lim_{x \to \pm\infty} \mathfrak{v}(x, \eta) = \mathfrak{v}_{\pm\infty} \tag{112}$$

im Sinne gleichmäßiger Konvergenz.

Beweis. Wegen (101) und (103) gilt allgemein

$$\mathfrak{v}_{\pm\infty} = \mathfrak{v}_\infty \pm \frac{E}{2t} \mathfrak{i} \pm \frac{\Gamma}{2t} \mathfrak{j}. \tag{113}$$

Bei Benutzung von (4), (5) und (12) folgt aus (102), (104), (105) und (113)

$$\mathfrak{v} - \mathfrak{v}_{+\infty} = \frac{1}{2t} \int_{\mathfrak{C}_0} \left\{ -v_n \left(\frac{t}{\pi} \operatorname{grad} F + \mathfrak{i} \right) + v_t \left[\frac{t}{\pi} \operatorname{grad} F + \mathfrak{i}, \mathfrak{k} \right] \right\} ds, \tag{114}$$

wobei der Gradient auf den Aufpunkt (x, y) bezogen ist. Bezeichnet (ξ, η) den Integrationspunkt auf $\mathfrak{C}_0$, so wird mit (106), (110) und (111)

$$\frac{t}{\pi} \operatorname{grad} F + \mathfrak{i} = 2 e^{-\frac{2\pi x}{t}} \{ \mathfrak{i}\, P(\xi, \eta; x, y) + \mathfrak{j}\, Q(\xi, \eta; x, y) \}. \tag{115}$$

Mit einer Zahl $a'_+ > a_+$ erhalten wir aus (97), (110) und (111) für alle (ξ, η) auf $\mathfrak{C}_0$, sowie für alle $x \geqq a'_+$ und alle y eines beliebigen abgeschlossenen Intervalls die gleichmäßige Beschränkung von $P(\xi, \eta; x, y)$ und $Q(\xi, \eta; x, y)$:

$$|P(\xi, \eta; x, y)| \leqq e^{\frac{2\pi a_+}{t}} \frac{1 + e^{\frac{2\pi(a_+ - a'_+)}{t}}}{\left(1 - e^{\frac{2\pi(a_+ - a'_+)}{t}}\right)^2},$$

$$|Q(\xi, \eta; x, y)| \leqq e^{\frac{2\pi a_+}{t}} \frac{1}{\left(1 - e^{\frac{2\pi(a_+ - a'_+)}{t}}\right)^2}.$$

Damit folgt aus (115) die Existenz des Grenzwertes

$$\lim_{x \to +\infty} \left(\frac{t}{\pi} \operatorname{grad} F + \mathfrak{i}\right) = 0 \tag{116}$$

im Sinne gleichmäßiger Konvergenz für alle (ξ, η) auf $\mathfrak{C}_0$ und alle y eines beliebigen abgeschlossenen Intervalles. Durch Vertauschung des Grenzübergangs $x \to +\infty$ mit der Integration über $\mathfrak{C}_0$ liefern (114) und (116) die Behauptung (112) für $x \to +\infty$. Entsprechend verläuft der Beweis für $x \to -\infty$.

Satz 7. *Es seien v_n und v_t zwei auf $\mathfrak{C}_0$ stetige Funktionen und $\mathfrak{v}_\infty$ eine vektorielle Konstante. Dann ist das Bestehen irgendeiner der beiden Integralgleichungen*

$$v_n + \frac{1}{\pi} \frac{\partial}{\partial n} \int\limits_{\mathfrak{C}_0} v_n F \, ds = 2(\mathfrak{n}, \mathfrak{v}_\infty) + \frac{1}{\pi} \frac{\partial}{\partial t} \int\limits_{\mathfrak{C}_0} v_t F \, ds, \tag{117}$$

$$v_t + \frac{1}{\pi} \frac{\partial}{\partial n} \int\limits_{\mathfrak{C}_0} v_t F \, ds = 2(\mathfrak{t}, \mathfrak{v}_\infty) - \frac{1}{\pi} \frac{\partial}{\partial t} \int\limits_{\mathfrak{C}_0} v_n F \, ds, \tag{118}$$

wobei F die Bedeutung (106) *hat, notwendig und hinreichend für die Existenz eines Vektorfeldes $\mathfrak{v}$ in $\mathfrak{A}$ und auf $\mathfrak{C}_k$, $k = 0, \pm 1, \pm 2 \ldots$, das außer den oben genannten Eigenschaften auf $\mathfrak{C}_0$ die Normalkomponente v_n und die Tangentialkomponente v_t besitzt und dessen Mittelwerte $\mathfrak{v}_{\pm\infty}$ der Gleichung* (103) *genügen. Zu jeder Kombination v_n, v_t, $\mathfrak{v}_\infty$, die* (117) *oder* (118) *genügt, gibt es genau ein derartiges Vektorfeld $\mathfrak{v}$; dieses wird durch die Gleichungen* (102) *bis* (106) *dargestellt.*

Beweis. Man kann zeigen, daß die Grenzwerte (25), (26), (31) und (32) unverändert gelten, wenn man $\ln \frac{1}{r}$ durch F (definiert durch (106)) und $\mathfrak{C}$ durch $\mathfrak{C}_0$ ersetzt; darüber hinaus können auch die in § 5 benutzten Existenz- und Eindeutigkeitssätze sinngemäß auf Gitterprofile übertragen werden [7]. Der Beweis verläuft daher weitgehend analog zu dem für Satz 2, und wir beschränken uns auf die Darstellung der Besonderheiten. Dazu zählt zunächst die Periodizitätsforderung (98), die wegen der Periodizität von (106) gesichert ist. Durch Einsetzen von (104) und (105) in (102) erhält man nach einer Umformung

$$\begin{aligned} \mathfrak{v} = \mathfrak{v}_\infty &\pm \frac{\mathfrak{i}}{2t} \int\limits_{\mathfrak{C}_0} v_n \, ds \pm \frac{\mathfrak{j}}{2t} \int\limits_{\mathfrak{C}_0} v_t \, ds - \\ &- \frac{1}{2t} \int\limits_{\mathfrak{C}_0} v_n \left(\frac{t}{\pi} \operatorname{grad} F \pm \mathfrak{i}\right) ds + \frac{1}{2t} \int\limits_{\mathfrak{C}_0} v_t \left[\frac{t}{\pi} \operatorname{grad} F \pm \mathfrak{i}, \mathfrak{k}\right| ds. \end{aligned} \tag{119}$$

Es wurde bereits gezeigt, daß $\frac{t}{\pi}\operatorname{grad} F + \mathrm{i}$ für alle (ξ, η) auf $\mathfrak{C}_0$ und alle y eines abgeschlossenen Intervalls gleichmäßig gegen Null konvergiert, wenn $x \to +\infty$ geht, so daß nach Multiplikation von (119) mit $e^{-\frac{2\pi x}{t}}$ auch der Grenzwert (99) für $x \to +\infty$ im Sinne gleichmäßiger Konvergenz existiert und den Wert Null hat. Das Entsprechende läßt sich für $x \to -\infty$ beweisen. Es bleibt schließlich zu zeigen, daß die Mittelwerte $\mathfrak{v}_{\pm\infty}$ von (119) Gleichung (103) genügen. Dazu beachten wir, daß für (119) als Element der zugrundegelegten Funktionenklasse der Satz 6 gilt, so daß wir die Mittelwerte $\mathfrak{v}_{\pm\infty}$ durch die Grenzübergänge $x \to \pm\infty$ bekommen. Da außerdem $\frac{t}{\pi}\operatorname{grad} F \quad \mathrm{i}$ für alle (ξ, η) auf $\mathfrak{C}_0$ und alle y eines abgeschlossenen Intervalles gleichmäßig gegen Null konvergiert, wenn $x \to -\infty$ strebt, gestattet (119) die Grenzübergänge $x \to \pm\infty$ auszuführen:

$$\mathfrak{v}_{\pm\infty} = \mathfrak{v}_{\infty} \pm \frac{\mathrm{i}}{2t} \int_{\mathfrak{C}_0} v_n \, ds \pm \frac{\mathrm{i}}{2t} \int_{\mathfrak{C}_0} v_t \, ds .$$

Damit ist (103) erfüllt.

Führt man den Kurvenparameter φ auf $\mathfrak{C}_0$ ein, so unterscheiden sich die erhaltenen Integralgleichungen von den in § 6 betrachteten nur durch die Kerne

$$(120) \qquad K(\varphi, \psi) = \frac{2\pi}{t} \frac{\dot{y}(\varphi)\,\mathfrak{Sin}\,\frac{2\pi[x(\varphi) - x(\psi)]}{t} - \dot{x}(\varphi)\sin\frac{2\pi[y(\varphi) - y(\psi)]}{t}}{\mathfrak{Cos}\,\frac{2\pi[x(\varphi) - x(\psi)]}{t} - \cos\frac{2\pi[y(\varphi) - y(\psi)]}{t}},$$

$$(121) \qquad L(\varphi, \psi) = \frac{2\pi}{t} \frac{\dot{x}(\varphi)\,\mathfrak{Sin}\,\frac{2\pi[x(\varphi) - x(\psi)]}{t} + \dot{y}(\varphi)\sin\frac{2\pi[y(\varphi) - y(\psi)]}{t}}{\mathfrak{Cos}\,\frac{2\pi[x(\varphi) - x(\psi)]}{t} - \cos\frac{2\pi[y(\varphi) - y(\psi)]}{t}}.$$

Definiert man $H(\varphi, \psi)$ wiederum durch (65), so kommen die zu (69) und (70) analogen Beziehungen

$$(122) \qquad K(\varphi, \psi) = 2\frac{\partial}{\partial\varphi} \operatorname{arc\,tg} \frac{\operatorname{tg}\frac{\pi[y(\varphi) - y(\psi)]}{t}}{\mathfrak{Tg}\,\frac{\pi[x(\varphi) - x(\psi)]}{t}},$$

$$(123) \qquad H(\varphi, \psi) = \frac{\partial}{\partial\varphi} \ln \frac{\mathfrak{Cos}\,\frac{2\pi[x(\varphi) - x(\psi)]}{t} - \cos\frac{2\pi[y(\varphi) - y(\psi)]}{t}}{\sin^2\frac{\varphi - \psi}{2}}$$

zustande, womit auch (71) und (72) Gültigkeit behalten. Die Grenzwerte (62) und (66) gelten ebenfalls unverändert. Damit kann das in § 7 beschriebene numerische Verfahren wörtlich übernommen werden, nur sind jetzt $\widetilde{K}_{\mu\nu}$ und $\widetilde{L}_{\mu\nu}$ Näherungen von (120) und (121).

Durch Anwendung des Impulssatzes auf die Berandung $\mathfrak{C}_0$ des Gitterprofils erhält man für die Gesamtkraft gleichfalls den Ausdruck (6), allerdings mit der veränderten Bedeutung (103) für $\mathfrak{v}_{\infty}$.

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(Eingegangen am 8. Januar 1960)

Poincaré and the Shortcomings of the Hamilton-Jacobi Method for Classical or Quantized Mechanics

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Communicated by C. Truesdell

Abstract

A great theorem was proven by H. Poincaré in celestial mechanics. It states that, in the most general problems of mechanics, the total energy of the system is the only well behaved first integral of the system, while other so-called integrals cannot be represented by uniform and convergent series. This very important result can be explained and visualized by comparison with standard methods of discussion, as, for example, the Hamilton-Jacobi procedure. The discussion shows that there are serious limitations to the use of this procedure, which collapses in the most general problems (Poincaré theorem) and can be used only for "almost separated" variables.

The Poincaré theorem appears to provide the distinction between determinism in mechanics and statistical mechanics according to Boltzmann.

1. Poincaré's Science and Hypothesis

Famous old authors are always worth rereading. Let us open Poincaré's book "Science and Hypothesis"[1] to Chapter VIII, where the author discusses the definition of "energy"; we find on page 158 (French edition, or page 121, English edition) a very curious problem stated in the following way:

"In the determinist hypothesis, the state of the universe is determined by an extremely great number n of parameters which I shall call $x_1, x_2, \ldots, x_n$. As soon as the values of these n parameters at any instant are known, their derivatives with respect to the time are likewise known and consequently the values of these same parameters at a preceding or subsequent instant can be calculated. In other words, these n parameters satisfy n differential equations of the first order.

"These equations admit of $n-1$ integrals, and consequently there are $n-1$ functions of $x_1, x_2, \ldots, x_n$, which remain constant. *If the we say there is something which remains constant,* we only utter a tautology. We should even be puzzled to say which among all our integrals should retain the name of energy."

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[1] Poincaré, H.: La Science et l'Hypothèse. Paris: Flammarion 1902; English translation, The Foundations of Science, Science and Hypothesis, The Value of Science, Science and Method. New York: The Science Press 1929.

The author presents there, in a paradoxical way, a very profound problem. In a private letter (December 1958), SCHRÖDINGER writes: "POINCARÉ is sort of pulling his reader's leg, in order to explain in the simplest way some very new and relevant points of view ... There is not just one constant of the motion but $(n-1)$. Which one of them is the energy? or rather, what function of those $(n-1)$? I believe these were points very worth making, and perhaps worth thinking about even now."

We intend to discuss the preceding statement, and we shall compare it with a very important theorem discovered by POINCARÉ[2] himself in his work on celestial mechanics, which contains a very pertinent answer to the problem. This will lead us into a discussion of the limits of accuracy in classical mechanics, and of the foundations of statistical mechanics. The problem was originally intended by POINCARÉ to apply to a general physical system, but his statement certainly cannot be used for a quantized structure since it does not include any possibility of quantum jumps. Hence we shall restrict the discussion to a *classical mechanical system*. We thus examine the following problem:

We consider a conservative mechanical system, with no dissipation (no friction, no passive resistance) nor regeneration (no external energy source, no negative (active) resistance). For such a model we easily define potential and kinetic energy, hence total energy too. This is the way all problems of classical mechanics are usually stated. POINCARÉ himself, when discussing a celestial problem, has no hesitation how to write down the total energy of the system.

There are N coordinates $x_1, x_2, \ldots, x_N$ and N corresponding momenta $p_1, \ldots, p_N$. This represents the n parameters of POINCARÉ:

$$n = 2N. \tag{1}$$

The initial state of the system at $t=0$ is defined when we give the initial values $x_1(0), \ldots, x_N(0)$, $p_1(0), \ldots, p_N(0)$ of the $2N$ parameters. The mathematician considers these initial values as "given", but the physicist asks "given by whom"? We must enquire how these initial values are measured and investigate the accuracy. This involves a complete change of point of view, and here we quote a remark of HADAMARD: "The question is not whether you make a very small change in the initial conditions, the real problem is whether this small initial change may not result in a very large change in the final results." In other words, the problem concerns the stability of the trajectories.

From $t=0$ on, the system moves on according to the laws of mechanics, and we may compute the coordinates $x_i(t)$ and momenta $p_i(t)$ at time t. These $2N$ quantities are defined in terms of one variable t and $(2N-1)$ constants, to be computed from the initial conditions. These $2N-1$ constants are the "first integrals" considered by POINCARÉ in his paradoxical statement in Science and Hypothesis.

2. Poincaré's great theorem on celestial mechanics

Let us now open a previous book[2] of POINCARÉ himself (1892), the first volume of his "Méthodes nouvelles de la Mécanique céleste", and read Chapter V, entitled:

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[2] POINCARÉ, H.: Méthodes Nouvelles de la Mécanique Céleste. Paris: Gauthier-Villars 1892–1899; reprinted by Dover Publications, New York.

Non-existence of uniform integrals. The whole chapter is devoted to the discussion of one fundamental theorem:

The canonical equations of celestial mechanics do not admit (except for some exceptional cases to be discussed separately) any analytical and uniform integral besides the energy integral.

This great Poincaré theorem contains the answer to the previous problem. It definitely proves that the $(n-1)$ integrals previously introduced by POINCARÉ have very different properties. The total energy is the only expression which is represented by a "well behaved" mathematical function, and all other so-called integrals are not analytical; they may have a very strange behavior, with discontinuities, and cannot be compared with the energy. The integrals representing momentum or moment of momentum are comprised in the other "exceptional cases" discussed by POINCARÉ.

For instance, POINCARÉ applies his theorem to the three body problem (vol. I, p. 253). Assuming the center of mass to remain at rest, there are 4 actual first integrals, corresponding to moment of momentum and energy; POINCARÉ'S theorem proves that no other well behaved first integral can be found, and it thus puts an end to one century of unsuccessful attempts.

Why did POINCARÉ choose not to mention this great theorem in his later book? The reason must be that, in Science and Hypothesis, he had in mind a much more general problem and was thinking of the behavior of any arbitrary physical system; but we already said that his statement certainly does not hold for quantum theory and that the problem must be restricted to classical mechanics only.

Our aim is to compare this theorem with the original statement of POINCARÉ, and in doing so, we have to present a short summary of the standard methods of Analytical Dynamics. We choose to follow an excellent textbook written by MAX BORN[3] on Atom Mechanics, where the classical methods are very precisely discussed for the problems of the "old" quantized mechanics, for which they represented the skeleton of the theory.

This text is also chosen because of a curious coincidence: after discussing very precisely the Hamilton-Jacobi method, and using it for computation on perturbed systems, M. BORN considers the great Poincaré theorem, which apparently contradicts the "general" perturbation methods of M. BORN and restricts their validity to the exceptional cases where the theorem does not apply. BORN just quotes two theorems of BRUNS and POINCARÉ (p. 292). Both theorems refer to the convergence of expansions obtained by the perturbation theory. The theorem of BRUNS (1884) states that points of absolute convergence or of divergence are equally dense. The perturbation procedure introduces non-analytic functions on one side and assumes the property of analyticity at another point: it thus contains internal contradictions. POINCARÉ'S theorem also refers to perturbation methods in celestial mechanics and goes further than BRUNS. BORN mentions these grave difficulties and does not elaborate further (he must have been too busy at the time with the "matrix mechanics" he was inventing, with HEISENBERG). He nevertheless adds some comments: "It thus was never possible to prove the stability of the solar system ... The perturbation methods used in celestial mechanics are not convergent, but only semi-convergent (POINCARÉ, vol. II, Chap. VIII). *We thus see that it is impossible, for purely theoretical reasons, to prove the absolute stability of atomic systems.*" Let us add that these semi-classical methods completely failed in the case of the helium atom.

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[3] BORN, MAX: Vorlesungen über Atommechanik, erster Band, collection „Struktur der Materie", vol. 2. Berlin: Springer 1925; English translation, M. BORN, The Mechanics of the Atom. London: Bell 1927. Also C. LANCZOS, Variational Principles of Mechanics. University of Toronto Press 1949.

For atomic structures, the solution was found with the invention of wave mechanics, which considers the old-fashioned quantized mechanics only as a first approximation of limited validity, as proved by the well known B. W. K. method.

We shall first remind the reader of the fundamental methods of classical mechanics, in order to be in a position to discuss the importance of POINCARÉ'S theorem and to examine the limitations it introduces upon the validity of classical methods.

3. The methods of analytical dynamics for separated variables

As stated in the preceding section, we refer the reader to BORN'S textbook for all details, and we shall be satisfied with a short summary of the general situation; quotations will be distinguished by a B (for BORN).

If we have *just one variable* ($N=1$), there is only one,

$$2N-1=1, \tag{2}$$

first integral, which will directly define the energy. *Assuming stability*, the motion is periodic, and we introduce an angular variable w, of which both x and p are periodic functions (period 2π)

$$w=\nu t+w_0; \tag{3}$$

during one period, w increases by one unit.

The corresponding momentum is the action variable

$$J=\oint p\,dx \tag{4}$$

taken for a full period. The energy depends only upon J

$$E=E(J), \tag{5}$$

and the equation of motion yields

$$\frac{dw}{dt}=\nu=\frac{\partial E}{\partial J} \tag{6}$$

where ν is the classical frequency. The action J was chosen for quantization, and the old quantized mechanics used the assumption

$$J=n\,h, \qquad n \text{ an integer}, \tag{7}$$

with a quantum frequency

$$\nu_q=\frac{\Delta E}{h}=\frac{\Delta E}{\Delta J} \qquad \Delta J=h. \tag{8}$$

The similarity of equations (6) and (8) was the basis of BOHR's *correspondence* principle. On the other hand, it was proven that any slow change of a parameter in the energy formula would leave J (and hence n) unchanged. This is the *principle of adiabatic invariance* (B., Chap. II, §§ 1—12) which applies *as long as the frequency ν does not become zero.*

These formulas can be easily generalized for a stable system with *separable variables*[4]. This is the case when the independent variables $x_1, \ldots, x_K$ can be chosen in such a way that each pair x_K, p_K obeys a separate equation of motion and is periodic with a frequency ν_k $\left(\text{the period being } \tau_k = \frac{1}{\nu_k}\right)$. We have (3), (4), (6), (7) as before (each one being given a subscript k), and the total energy, instead of equation (5), is given by an expression

$$E = E(J_1, J_2, \ldots, J_N) \tag{9}$$

containing only the action variables J. Such a stable system is called *multiperiodic*. If we consider a quantity represented by a function of the variables x_k, p_k, it can be rewritten as a function of the angular variables w_k

$$F(w_1, w_2, \ldots, w_N),$$

and it exhibits period 1 in each of the w_K variables. These variables themselves are linear functions of time (equation (3)) and the function F can be expanded in a multiple Fourier series

$$F = \sum_m \underset{m}{C}\, e^{2\pi i (m\cdot w)} = \sum_m \underset{m}{C}\, e^{2\pi i t (m\cdot\nu) + 2\pi i (m\cdot w_0)}, \tag{10}$$

where m represents a set of integers $m_1, m_2, \ldots, m_N$, and $(m\cdot w)$ or $(m\cdot\nu)$ are scalar products

$$(m\cdot w) = \sum_k m_k \cdot w_k \qquad (m\cdot\nu) = \sum_k (m_k\cdot\nu_k).$$

The system is *non-degenerate* if *no relation*

$$(m\cdot\nu) = 0, \qquad m \text{ integers}, \tag{11}$$

can be found; otherwise it is called *degenerate*. In such a case, the number of independent angular variables is reduced. If we have n relations (11), the number of independent frequencies is $N - n$. The system is completely degenerate when there remains only one angular variable and one frequency ν. This is the case for instance for Kepler ellipses or for Lissajoux figures.

The condition (11) for non-degeneracy is very restrictive. It means that all frequencies are incommensurable, so that there exists no zero combination

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[4] BORN, M.: Chapter II, paragraphs 14—17. Beware of the fact that BORN uses different notations

	Born	here
frequency	ν_k	ν_k
period	ω_k	τ_k
integer	τ_k	m_k

frequency among them. On the other hand, some *quasi-periods* τ_q (POINCARÉ) can be found, that satisfy relations

$$\tau_q = m_1 \tau_1 + \varepsilon_1 = m_2 \tau_2 + \varepsilon_2 \cdots = m_N \tau_N + \varepsilon_N \tag{12}$$

with errors $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_N$ as small as desired. If all the ε were zero for a certain choice of the m_K's, the τ would really be a common period and the system would completely degenerate, since equation (12) could be written

$$\frac{\nu_1}{m_1} = \frac{\nu_2}{m_2} \cdots \frac{\nu_N}{m_N} = \nu = \frac{1}{\tau_q}, \tag{13}$$

which represents $N-1$ rational relations between the original periods ν_K.

In the old quantized mechanics, the J action variables corresponding to the w's were quantized, according to equation (7). These J's were *adiabatic invariants* for any slow change of a parameter in the equations, *so long as the non-degeneracy was maintained and no condition* (11) would occur between the frequencies slowly modified by the change of the parameter. This condition of no degeneracy is so restrictive as to make the *invariance very seldom applicable.*

It may however happen that the change of the parameter does not establish any coupling between the separate variables, instead maintaining complete independence of their equations. If such were the case, the occasional degeneracy would no longer be an obstacle.

Coming back to our equations with separated variables, we note that the system has $2N-1$ integrals, namely the N action variables J_K (which define the energy) and the $N-1$ phase differences $(w_{k0} - w_{10})$.

These are the constants considered in Section 1. Their respective importance and their general behavior will be discussed later.

4. Non separable variables. Hamilton-Jacobi procedure

When variables cannot be separated, the problem becomes much more involved. One may try a formal solution, based on Hamilton-Jacobi equations, or one will wish to be more practical and use a method of approximation.

We restrict our discussion to the case of *stable problems*, when no particle may go to infinity.

The *Hamilton-Jacobi* equations yield a possibility of finding angular variables w_k and the corresponding action variables J_K which formally satisfy equations very similar to those of the preceding section. These variables are often called Delaunay variables (see LANCZOS). A typical feature results from the fact that this procedure now involves a "contact transformation" instead of a "point transformation". The Poincaré theorem of Section 2 will play a distinctive role in emphasizing the difficulties of this general theory. The classical application of the Hamilton-Jacobi method is given by BORN (B., §§ 5–8, 15), and the discussion proves that difficulties arise whenever a degeneracy condition (11) obtains. This means that the definition of the w, J variables may be materially altered at points of degeneracy.

Such a restriction was easily eliminated in a problem with separated variables, because there was no coupling possible between different variables. Here one must admit that this sort of coupling does exist in general, and that occasional degeneracy must cause trouble.

At any rate, the Hamilton-Jacobi procedure has never been a way to solve a practical problem. It is very useful for general theorems and discussions, but not for actual solution.

5. Successive approximations

The only available method for the solution of a special problem is the *method of successive approximations* (B., Sections 18, 34—35 and 40—44). In introducing the method, it is necessary to start with the case of degeneracy (B., Section 18). We assume an unperturbed energy function given by the Hamiltonian H_0 and a small perturbation λH_1 (λ small),

$$H = H_0 + \lambda H_1. \tag{14}$$

The problem is supposed to have been already solved for the unperturbed Hamiltonian H_0, and it may yield a partly degenerate solution. The small perturbation is usually enough to remove the degeneracy. For instance, the initial problem, being n times degenerate, depends only upon $N-n$ angular variables (see Section 3, equation (11)) and obtains that many actual frequencies, while n frequencies are zero. When the degeneracy is lifted, the $N-n$ original frequencies are modified by an amount proportional to λ, while the n additional frequencies take small non-zero values, of the order of λ. Each perturbation function H_1 will react differently on the n degenerate degrees of freedom, and result in a different choice for these additional w and J variables. Hence we see directly how a *point of degeneracy* will be a point for *possible sudden changes in the definition of angular and action variables.*

Let us consider a non-degenerate system characterized by a perturbation term λH_1, and the corresponding w, J variables. Then, let us decrease λ to zero and produce degeneracy; finally we introduce another perturbation $\lambda' H_1'$, which will result in a different choice of angular variables w' and action variables J'. The original J's were adiabatic invariants for a slow change of λ; for $\lambda=0$ we have a sudden change in the J values, and the new J's are adiabatic invariants for a slow variation of λ'.

A well known example in atomic models is represented by an atom first perturbed by a magnetic field (ZEEMAN effect, B., § 34) and then by an electric field (STARK, B., § 35).

To be more specific, let us assume a system with 2 degrees of freedom which both yield the same frequency ν_0 in the unperturbed problem. This may be a Kepler ellipse where r and ϑ have the same frequency ν_0. The same ellipse may also be represented in x, y coordinates, and they have again the same frequency ν_0. The first perturbation λH_1 may separate r and ϑ, giving them frequencies ν_r and ν_ϑ, both slightly different from ν_0; the difference $\nu_r - \nu_\vartheta$ represents the frequency which is zero in the degenerate case and becomes a small quantity of the order of λ in the perturbed problem. The second perturbation

$\lambda' H'$ may be a field in the x direction, acting differently on x or y and producing frequencies ν_x and ν_y. In both cases, action variables are easily found. The transformation $\lambda H_1 \to \lambda' H_1'$ exhibits

$$(15)\qquad \begin{array}{ll} \text{continuity of the frequencies} & \nu_r, \nu_\vartheta \to \nu_0 \to \nu_x, \nu_y \\ \text{discontinuity in the action variables} & J_r, J_\vartheta \neq J_x, J_y. \end{array}$$

The formulas (6) for the frequencies are always valid despite the discontinuous change in the J variables:

$$(16)\qquad \begin{aligned} \nu_r &= \frac{\partial H_0}{\partial J_r} + \lambda \frac{\partial H_1}{\partial J_r} = \nu_0 + \lambda \frac{\partial H_1}{\partial J_r} \\ \nu_\vartheta &= \nu_0 + \lambda \frac{\partial H_1}{\partial J_\vartheta} \\ \nu_x &= \nu_0 + \lambda' \frac{\partial H_1'}{\partial J_x} \\ \nu_y &= \nu_0 + \lambda' \frac{\partial H_1'}{\partial J_y}\,. \end{aligned}$$

We have oversimplified the problem in order to show more clearly the fundamental mechanism involved. Let us generalize and assume a degeneracy of the type

$$(17)\qquad m_1 \nu_1 - m_2 \nu_2 = 0, \qquad m_1 > 0 \quad m_2 > 0 \text{ integers},$$

hence with the periods

$$\tau_1 = m_1 \tau, \qquad \tau_2 = m_2 \tau$$

$$\tau = \frac{1}{\nu} = \frac{1}{m_1 \nu_1} = \frac{1}{m_2 \nu_2}.$$

The action integral J at degeneracy is related to J_1 J_2 before degeneracy by the formula

$$(18)\qquad J = \oint_\tau (p_1\, d x_1 + p_2\, d x_2) = \frac{1}{m_1} J_1 + \frac{1}{m_2} J_2.$$

This relation shows the J discontinuity.

6. Approximations for non-degenerate systems

The preceding problem represents actually the general case. When we start from a non-degenerate system and modify even very slightly any parameter, we immediately pass through a very large number of degenerate situations. *Non*-degeneracy requires a set of incommensurable frequencies. We already noticed, in Section 3, that adiabatic invariance of the action variables could be proven only so long as non-degeneracy was obtained or separation of variables maintained, preventing any coupling between different groups of variables. In the general case, we cannot assume complete separation, and the slightest perturbation destroys the non-degeneracy, hence the definition of angle and action variables becomes hopeless. It may have to be changed at each point of degeneracy, and these points lie infinitely close to each other. This essential difficulty is easily recognized in the general discussion sketched by BORN (B., § 41). Assuming a Hamiltonian of the type

$$H = H_0 + \lambda H_1 + \lambda^2 H_2 \ldots \qquad \lambda \text{ small} \tag{19}$$

where H_0 is the unperturbed Hamiltonian and $H_1, H_2, \ldots$ are perturbation functions, the resolution requires the formation of a transformation function

$$S = S_0 + \lambda S_1 + \lambda^2 S_2 \ldots, \tag{20}$$

and the successive terms S_1, S_2 are found to be of the general type (B., p. 291)

$$S_n = \sum_m{}' \frac{1}{2\pi i} \frac{A_m}{(m \cdot \nu_0)} e^{2\pi i (m \cdot w)} \quad \text{(with no constant term)}, \tag{21}$$

a formula where $(m \cdot \nu_0)$ and $(m \cdot w)$ represent scalar products defined for equation (10), while the A_m coefficient is computed from $H_1, H_2 \ldots$ perturbation functions. If the system is originally non-degenerate, then no denominator $(m \cdot \nu_0)$ can be identically zero, and BORN assumes that the expansion is convergent. The case is not that simple since a great many of them will be as nearly zero as desired (or rather, as *not* desired).

These extremely small denominators may make the sum divergent; and we must add that the smallest perturbation would make some of these $(m \cdot \nu)$ actually zero and S_n infinite. The perturbed action variables J are defined in terms of the S function and thus open to any kind of irregular behavior and discontinuities. The frequencies are again continuous. This situation will help us understand the exact meaning of the great Poincaré theorem.

In the preceding discussion, it was assumed that degeneracy or non-degeneracy was directly controlled by the nature of the Hamiltonian. Let us remember that *initial conditions* of the motion also play a decisive role; for instance, the frequency of a physical pendulum is a function of amplitude; if we have a pendulum oscillating along x and y, with different non-linear restoring forces in both directions, we easily see that the frequencies ν_x and ν_y can be made commensurable or incommensurable by very small changes of the initial conditions. Hence initial conditions also control degeneracy. This point will be of importance for later discussion.

7. Poincaré's great theorem again

We must now come back to POINCARÉ's great theorem, which we quoted in Section 2 without examining carefully its meaning and importance. The importance of this theorem is well recognized by astronomers, but mathematicians or physicists often overlook it; its implications reach very far and shatter many current ideas in classical mechanics. Let us first explain the exact statement of POINCARÉ. He considers a mechanical system obeying the canonical equations

$$\frac{dx_i}{dt} = \frac{\partial F}{\partial y_i}, \qquad \frac{dy_i}{dt} = -\frac{\partial F}{\partial x_i}, \tag{22}$$

where $x_1, \ldots, x_i, \ldots, x_N$ are the variables and $y_1, \ldots, y_i, \ldots, y_N$ are the corresponding moments, while $F(x_i, y_i)$ is the total energy of the system. The problem of POINCARÉ is the three-body problem of celestial mechanics, and the energy F is supposed to be expanded in power series of a certain parameter μ

$$F = F_0 + \mu F_1 + \mu^2 F_2 + \cdots. \tag{23}$$

This represents a typical case of perturbation, and the μ expansion of equation (23) corresponds to the λ expansion of our previous equation (19), since H or F both represent the total energy. What is different in POINCARÉ's case is the choice of the x_i, y_i variables.

In the unperturbed problem, it is assumed that F_0 depends upon the N variables x_i but not upon the y_i's. The perturbation functions $F_1, F_2 \ldots$ depend upon both x_i and y_i variables. Furthermore, the whole problem is multiperiodic (hence stable), and here is how it happens:

The mechanical equations (22) give, in the unperturbed problem,

$$x_i = x_{i0} \text{ constant}; \qquad y_i = -\frac{\partial F}{\partial x_i} t + y_{i0}, \tag{24}$$

and the original coordinates of the system (POINCARÉ, *loc. cit.*, pp. 22—32) are multiperiodic functions of the y_i's. Hence the y_i (momentum variables) play the role of our previous w_i (angular variables) while the x_i variables replace the previous action variables. Finally, the situation is exactly similar to the previous problem, with just an interchange of variables and momenta, and such an interchange is of no consequence whatsoever.

Accepting y_i as representing an angular variable w_i, we see that

$$-\frac{\partial F}{\partial x_i} = \nu_i \tag{25}$$

is the corresponding frequency. (There is, however, a different use of the 2π factors in the treatments of BORN or POINCARÉ, but this again is immaterial.)

The system is multiperiodic with respect to the y_i variables, and this is also true for the F_1, F_2 perturbation terms.

With these assumptions, POINCARÉ explains the precise meaning of his theorem:

Let φ be an analytic and uniform function of x, y, μ which must also be periodic with respect to the y variables. Let this function φ be supposed to be analytic and uniform for all values of the y's, for small values of μ, and for x's contained in a certain domain D. Under such conditions, the function φ can be expanded

$$\varphi = \varphi_0 + \mu \varphi_1 + \mu^2 \varphi_2 + \cdots \tag{26}$$

with $\varphi_0, \varphi_1, \varphi_2, \ldots$ uniform in x, y and periodic in the y's. POINCARÉ then states: *Except for some special cases to be discussed later, such a function cannot be an integral of the system of equations* (22). *Only the total energy F has such properties.*

This theorem invalidates the system of approximations developed by BORN, who tried to find expansions of type (26) for his action variables J in a perturbed problem by using a transformation function S also expanded in a similar fashion. The S and J's cannot be well behaved analytic functions. This is the very remarkable consequence of POINCARÉ's theorem.

We do not intend to repeat here the details of the proof given by POINCARÉ, but we shall stress a few points of importance, since they explain clearly that all the trouble comes from points of *degeneracy*, which also give difficulties in BORN'S discussion. This *old theorem* of POINCARÉ proves that these difficulties are much more serious than BORN imagined and that they build up an almost impassable roadblock.

8. The role of degeneracy conditions in Poincaré's theorem

The clue, in POINCARÉ'S discussion, is to consider, for each stage of approximation, the behavior of the function φ at the points of degeneracy. These points are defined by conditions

$$\sum m_i \frac{\partial F_0}{\partial x_i} = \sum m_i \nu_i = 0, \quad m_i \text{ integers}, \tag{27}$$

according to (25). And POINCARÉ notes that, in any domain of the x_i's, even the smallest domain you can imagine, there is always an infinity of such points of commensurability (or degeneracy; see, for instance, POINCARÉ, p. 239, bottom of the page).

The perturbation function F_1 is expanded in multiple Fourier series (POINCARÉ, pp. 236—238),

$$F_1 = \sum B_m e^{2\pi i (m \cdot y)}, \tag{28}$$

a formula similar to our equation (10). The general case refers to a perturbation F_1 that may contain all the B_m terms. In such a case, the proof shows that the function φ cannot exist. When some of the B coefficients are zero, the discussion runs differently (POINCARÉ, pp. 240—245) since there is a possibility of secular perturbation. For certain classes of problems, it is still possible to prove the great POINCARÉ theorem, but for another group of problems, there appears a possibility of finding a certain limited number of well behaved first integrals, in addition to the energy integral. These are the exceptional cases announced by POINCARÉ in the statement of his theorem. These exceptional cases practically correspond to problems where a limited number of sets of variables can be separated away.

A typical example corresponds to a problem with conservation of the total momentum or moment of momentum.

There are many other interesting remarks to be found in POINCARÉ'S remarkable book, and they apply not only to celestial mechanics but to general mechanical problems. In his third volume (Chap. 28—31), POINCARÉ defines *periodic solutions of the second kind*, which correspond to another example of degeneracy: some trajectories exhibit a period τ, but other trajectories, very close to the first ones, may not be periodic at all or may have a much longer period $n\,\tau$, a multiple of the original τ. An example of these conditions was found in the problem of degeneracy discussed in equations (17), (18).

We can say, in conclusion of these two sections, that the very fundamental properties discovered by POINCARÉ in his great theorem are provoked by degeneracy conditions when the Hamiltonian gives rise to coupling between the different sets of variables.

9. Degeneracy conditions and the possibility of finding a Hamilton-Jacobi transformation function

We shall now generalize the preceding remarks and show that the Hamilton-Jacobi transformation function S itself may, in general, be very difficult or even impossible to obtain. We shall prove that the Hamilton-Jacobi method works wonderfully for separated variables but usually becomes impracticable when the variables are not separable.

We start from the following formulas[5], which can be taken as a definition of the transformation function S (S was called F in my book on tensors) of equation (20):

$$\frac{\partial S}{\partial q_k} = p_k; \quad \text{hence} \quad dS = \sum \frac{\partial S}{\partial q_k} dq_k = \sum p_k\, dq_k. \tag{29}$$

We thus can write

$$S = \int_{q_{k0}}^{q_k} p_k\, dq_k = \int_{q_{k0}}^{q_k} p_k\, \dot{q}_k\, dt = \int_{q_{k0}}^{q_k} 2E_{\text{kin}}\, dt. \tag{30}$$

The kinetic energy E_{kin} was called T in my book. The integral is taken along a natural trajectory running from an initial point q_{k0} at $t=0$ to a final point q_k at t. This integral is completely defined when we specify the total energy E of the system under consideration; once we know E, and we give the position $q_k(t)$ at time t, then the potential energy $V(q_k)$ is known, and the difference $E-V$ gives E_{kin}. This, in turn, determines the velocity along the trajectory. The total time interval required for a run from q_{k0} to q_k need not be specified separately. For further discussion, see my book[5] on tensors.

We wish to consider a stable system, with no particle running to infinity; such a system will be multiperiodic with no secular term. We thus *assume* a *multiperiodic* solution, where all quantities (E_{kin}, for instance) can be expanded in multiple Fourier series of type (10):

$$E_{\text{kin}} = \sum_m C\, e^{2\pi i t (m\cdot\nu) + 2\pi i (m\cdot w_0)}. \tag{31}$$

The integration (30) yields[6]

$$S = 2 \sum \frac{C_m}{2\pi i (m\cdot\nu)}\, e^{2\pi i t (m\cdot\nu) + 2\pi i (m\cdot w_0)}. \tag{32}$$

The question now is *what happens when the denominator* $(m\cdot\nu)$ *is zero* (complete *degeneracy*) or even simply very small (as small as desired, approximate degeneracy)?

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[5] Brillouin, L.: Les tenseurs en mécanique et en élasticité. Paris: Masson 1938; see Chap. VIII, paragraphs 12—16, especially p. 171. These definitions were systematically used by L. Boltzmann in his lectures on mechanics.

[6] Let us specify clearly the difference between equations (21) and (32). Equation (21) contains the "unperturbed" frequencies ν_0 corresponding to the H_0 Hamiltonian. In the other equation (32) the formula works with the final ν frequencies corresponding to the full problem ruled by the complete Hamiltonian H, including the so-called perturbation. Either one of these formulas can be used and may be more convenient for discussion.

It is at once obvious that such conditions are critical and may mean divergence of the multiple Fourier series, hence impossiblity of building up the function S. We must here consider two distinct problems: separated or non-separable variables.

When *variables can be separated*, there is absolutely no coupling between a q_k variable and another q_1 one. There exist *no cross-terms* either in the kinetic or in the potential energy. As a result, there must be no cross-terms in the Fourier expansion (31) and no troublesome terms in equation (32):

$$(33)\quad S = S_1(q_1) + S_2(q_2) + \cdots S_k(q_k) + \cdots S_n(q_k) + \sum_k n_k J_k \qquad (n \text{ integers}).$$

The transformation function is a sum of separate functions of each variable, but a function $S_k(q_k)$ is only defined modulo J_k for the simple reason that we may use, from the initial q_{k0} to the final q_k position, an arbitrary trajectory making m_k loops (each of frequency ν_k). Each S_k can be separately expanded in a single Fourier series

$$(34)\qquad S = \sum_k \left\langle \sum_{m_k} C_{m_k} e^{2\pi i m_k \nu_k t + 2\pi i m_k w_{0k}} + n_k J_k \right\rangle$$

where we recognize at first sight the absence of any term in

$$(m \cdot \nu)\, t = (\sum m_k \nu_k)\, t\,.$$

The problem may be non-linear in each separate variable, but cross-terms are not allowed. As noted in Section 3, the total energy is a function of the J_k action variables (equation (9)), which are themselves analytic.

If the variables cannot be separated, the situation is completely different. POINCARÉ's theorem proves that in the most general case the J_k quantities *cannot be analytic*, and our formula (32) shows that *the S function diverges*. The Hamilton-Jacobi procedure collapses. This is what happens if we make no special assumption about the Hamiltonian.

Many authors, including M. BORN, overlooked the non-analyticity of the J's in the general case, and assumed that one could still write a formula of type (9), giving the energy as a function of the J's. It would however be completely meaningless to express a well behaved expression, the energy, as a function of non-analytic quantities, the J's. The whole method simply does not work in the most general problems, when no restriction is used to limit the values of the C_m coefficients.

10. Sketch of a discussion of the possibilities of convergence for non-separated variables

We found that the Hamilton-Jacobi method works perfectly for separated variables, but does not work, in general, for non-separated variables: where is the exact limit between these two extremes? We shall find problems of "approximate separation" where a solution is still possible, provided certain conditions are fulfilled.

Let us consider again a perturbation problem, with a Hamiltonian (14)

$$H = H_0 + \lambda H_1 \tag{14}$$

where H_0 is the unperturbed system with separation of variables and H_1 perturbation with coupling terms between variables.

We may discuss this problem according to the perturbation method of Sections 5 and 6. We look for an expansion of our function S (see equation (20))

$$S = S_0 + \lambda S_1 + \lambda^2 S_2 \ldots \tag{20}$$

where S_0 corresponds to separation and is represented by equation (33), while $S_1, S_2 \ldots$ contain the whole series of terms as in equation (21):

$$S_1 = \sum_m \frac{A_m}{2\pi i (m \cdot \nu_0)} e^{2\pi i (m \cdot w)} \tag{21}$$

where ν_0 represents the unpeturbred frequencies[6]. We discussed such formulas at the end of Section 6, and we followed Born's distinction between (A) non-degenerate problems, where no $(m \cdot \nu_0) = 0$ relation can be found, (B) degenerate problems, with some $(m \cdot \nu_0) = 0$ relations

This distinction, however, looks rather artificial.

In a non-degenerate situation (A), we still have an infinite number of infinitely small $(m \cdot \nu_0)$ denominators and the corresponding numerators A_m must be extremely small if divergence is to be avoided. In a degenerate case (B), as soon as one denominator $(m \cdot \nu_0)$ is zero, there is an infinite number of other $(p\, m \cdot \nu_0)$ zero denominators (p an integer), and all the corresponding numerators must be zero. The distinction is really superficial.

The actual problem is concerned with the convergence of the series of coefficients $\frac{A_m}{(m \cdot \nu_0)}$, which should be discussed. The multiple Fourier series for S_1, $S_2 \ldots$ are all of the same type (21) where m stands for the set of indices $m_1, m_2, \ldots, m_N$. These series converge only when the series of the coefficients $\frac{A_m}{(m \cdot \nu_0)}$ are absolutely convergent. If this be the case, the S function can be constructed, and the Hamilton-Jacobi method works correctly.

If the absolute convergence of the multiple series of the coefficients $\frac{A_m}{(m \cdot \nu_0)}$ cannot be proven, the multiple Fourier series for S diverges, the Hamilton-Jacobi method collapses and we are in the "general situation" described by Poincaré. In order to achieve convergence, it is obviously necessary that the A_m coefficient become zero as $(m \cdot \nu_0)^\alpha$ with an exponent $\alpha \geqq 1$; it is also required that the A_m's themselves decrease fast enough for very large values of the $m_1, m_2 \ldots m_N$ indices. When such conditions are fulfilled, the Hamilton-Jacobi method can be used, and it extends far beyond the case of separated variables. These examples may be called problems of approximate separation.

11. Discussion of a simple example with two variables; degeneracy means instability or resonance

Degeneracy occurs when one frequency is zero, or when a condition of commensurability exists between the original frequencies. If the zero frequency corresponds to one of the variables of the motion, then it usually means a situation of instability. Such is the case for a rigid pendulum oscillating with very large amplitudes and reaching the top with zero velocity, whence it may fall back either to the right or to the left. The condition of commensurability will be shown to correspond to *resonance* between two or more degrees of freedom.

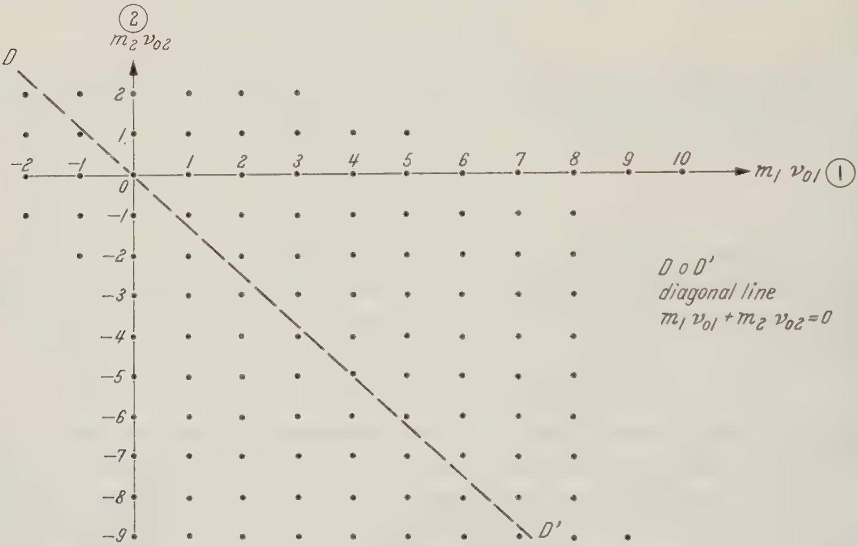

Fig. 1

Let us sketch the problem for a system with two variables, hence two unperturbed frequencies ν_{01} and ν_{02}. Figure 1 visualizes the general situation: in a plane, we use the $m_1\nu_{01}$ as abscissae and $m_2\nu_{02}$ as ordinates (m_1, m_2 integers). Degeneracy holds when one of the points of the rectangular lattice happens to fall upon the second diagonal DOD'.

$$(m \cdot \nu) = m_1\nu_{01} + m_2\nu_{02} = 0 \quad \text{or} \quad m_1\nu_{01} = -m_2\nu_{02}. \tag{35}$$

On our diagram, this is almost the case for $m_1=4$, $m_2=-5$. The numerators $A_{m_1m_2}$ must be zero along the diagonal line DOD'. When the variables are separated, the $A_{m_1m_2}$ are different from zero only along the axis 1 and the axis 2 and remain zero everywhere else in the plane.

Approximate separation holds when the $A_{m_1m_2}$ become zero along DOD' and decrease suitably at large distance.

Let us now consider the case of *divergence* and try to understand the physical meaning of the difficulty encountered. Condition (35) means *resonance*: the m_1 harmonic of ν_{01} coincides with the m_2 harmonic of ν_{02}. This certainly must cause trouble, since it is well known that resonance leads to infinite terms when no damping is available.

In many problems with no damping terms (neither positive nor negative), it happens that coupling between two oscillations at resonance may destroy resonance and give rise to two distinct frequencies. This is, in a different language, the well known result that any perturbation (coupling) applied upon a degenerate system (having internal resonance) will lift the degeneracy (destroy resonance).

When two periods $m_1 \nu_{01}$ and $-m_2 \nu_{02}$ happen to be almost equal in the unperturbed problem, there is no complication so long as the variables remain completely separated and uncoupled. But when we introduce intercoupling through the λH_1 perturbation term (equation (14)), we must be prepared for the possibility of a large finite energy transfer from ν_{01} to ν_{02} through internal resonance, even when the λH_1 coupling is extremely small. This is another way of explaining the difficulties discussed in Section 5. It would be interesting to discuss carefully what happens in a variety of cases of internal resonance, for different types of non-linear problems with non-linear coupling, and to discover progressively the type of interactions able to take place. This general program is intended for further investigation.

Let us now refer again to Note 6. We find difficulties when there appears some resonance between the *unperturbed frequencies* $\nu_{01}, \nu_{02}, \ldots, \nu_{0N}$, but we do not know whether any such resonance can take place between the final frequencies $\nu_1, \nu_2, \ldots, \nu_N$, defined in Section 9. There are indications that exact resonance between the final (actual) frequencies ν may not possibly occur, and this again is a point requiring careful discussion.

12. Some general conclusions. Determinism versus statistical mechanics

We discussed problems of classical mechanics, assuming that the model under consideration was stable, with no part of it escaping to infinity. The motion was considered as represented by a multiperiodic solution. In such a problem, the Hamilton-Jacobi method can be used only when the expansion of the transformation function S is convergent. This condition is automatically satisfied for separated variables, and can be satisfied for "almost separated" variables. In the general case the S expansion diverges, and POINCARÉ's great theorem definitely proves that the whole method is worthless.

The problem can be examined from two different angles, giving two distinct and even opposing points of view: the pure mathematical aspect first, and the viewpoint of a physicist, second.

The preceding considerations were based on a purely *mathematical* discussion. It was assumed that a trajectory could be defined by "given" initial conditions, the values of the coordinates and momenta at time $t=0$ being exactly known, with no possible error. It was also assumed that the laws of motion were absolutely correct, with no deviation whatsoever. These two assumptions led to the consideration of a single trajectory considered as an infinitely thin mathematical curve. The periods of the multiperiodic motion were supposed to be exactly known, and questions were raised about their commensurability. The answers to the problem were distinguished as "general" or "degenerate" whether the periods were incommensurable or commensurable.

To all these definitions, the *physicist* raises some very serious objections. His position is the following:

A. Initial conditions are not "given"; they must be measured and observed. This means that the coordinates x_i and momenta p_i are known within errors Δx_i and Δp_i.

B. The laws of motion are no better known. They result from experiments, and they summarize a large number of previous observations. The simple laws selected for this representation must be considered only as a first approximation. The famous laws of NEWTON'S mechanics are no exception, and the proof of this statement has been repeatedly given: these laws had to be amended and corrected in many instances, for Relativity, for Quantum Theory, *etc.* ...

From these premises it is easy to draw some obvious conclusions, which were already stated by E. BOREL, M. BORN and the present author (see the references in footnotes 7 and 8) and which were sketched briefly at the end of Section 1 of this paper.

a) The notion of an *irrational* number is *unphysical*. It is impossible to prove that a physical quantity is rational or irrational.

b) The notion of *commensurability* is *unphysical*, when the quantities to be compared are known only to within some experimental errors.

c) It is impossible to study the properties of one single (mathematical) trajectory. The physicist knows only *bundles of trajectories*, corresponding to slightly different initial conditions. E. BOREL, for instance, computed that a displacement of 1 cm, on a mass of 1 gram, located somewhere in a not too distant star (say, Sirius) would change the gravitational field on the earth by a fraction 10^{-100}. The present author went further[8] and proved that any information obtained from an experiment must be paid for by a corresponding increase of entropy in the measuring device: infinite accuracy would cost an infinite amount of entropy increase and require an infinite energy! This is absolutely unthinkable.

d) Let us simplify the problem, and assume that the laws of mechanics are rigorous, while experimental errors appear only in the determination of initial conditions. In the bundle of trajectories defined by these conditions, some may be "non-degenerate" while others may "degenerate". The bundle may soon explode, be divided in a variety of smaller bundles forging ahead in different directions. This is the case for a model corresponding to the kinetic theory of gases. BOREL computes that errors of 10^{-100} on initial conditions will enable one to predict molecular collisions for a split second and no more. It is not only "very difficult", but actually impossible to predict exactly the future behavior of such a model. The present considerations lead directly to BOLTZMANN'S statistical mechanics and the so-called "ergodic" theorem.

The problem is of such importance that it requires a few more words of explanation. Recalling the results of Section 10, we must distinguish between three different possibilities:

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[7] BOREL, E.: Introduction géométrique à quelques théories physiques, p. 94. Paris: Gauthier-Villars 1914. — BORN, M.: Phil. Quart. **3**, 139 (1953). — Kgl. Danske Videnkab. Selskab., Mat-fys. **30**, No. 2 (1955). — J. de Physique **20**, 43 (1959).

[8] BRILLOUIN, L.: Information and Control **1**, 2 (1957). — Nature **183**, 501 (1959). — Annals of Physics **3**, 243 (1958).

1. Separated variables. We must use the variables w_k and J_k introduced in Section 3. The J_k retain constant values through the bundle and are defined within errors ΔJ_k that depend only upon the errors made about initial conditions. These ΔJ_k remain constant in course of time. The problem was discussed in great detail elsewhere[9]. The angle variables w_k exhibit a very different behavior. The error Δw_k increases steadily in course of time, and after a while it may become larger than 2π, which means that we know no longer what this angle may be.

In such problems, the J_k are well behaved functions, but the angles w_k are not. Of course, predictions about the w_k values may be good for a certain period of time, right at the start; or even (if initial conditions are very exactly defined) the predictions may be correct for a rather long interval of time (as is the case in Astronomical problems), but in the long run the uncertainty over the phase angles must always become dominant.

2. Almost separated variables. Similar conclusions hold but for the fact that the J_k w_k variables now result from a contact transformation based on the function S instead of a point transformation.

3. General case (Poincaré). No S function can be constructed, and the total energy is the only well behaved function in a system of coordinates where the center of mass is at rest.

For separated or almost separated variables, we obtain *determinism* at the beginning, and after a certain time interval the angles w_k are no longer observable, but the momenta J_k are still well behaved integrals of the motion.

In the general case, the total energy is the only quantity about which predictions can be made. (See the reference in footnote 9, Chap. vi, p. 214.)

e) We may choose another simplified alternative and assume initial conditions to be rigorously "given" but fix our attention on the fact that the laws of mechanics are only a first approximation. These classical laws correspond to an "unperturbed" Hamilton function H_0, which may be completed by some unknown small perturbation λH_1. What is the role played by this perturbation? This is practically the question answered by Poincaré in the famous theorem quoted in Section 2 and 7. In the general case, the total energy comes out as the only well behaved analytic solution of the initial conditions. It appears that the Poincaré theorem contains the justification of Boltzmann's statistical mechanics, which should apply when (and only when) the total energy of the system actually remains the only well behaved first integral.

f) All these approximations must be carefully weighed and discussed. For some problems (kinetic theory of matter) they make predictions almost impossible. For other types of problems (astronomy) the situation is not so hopeless. The fundamental question to be discussed is the *stability of the solution* for small changes in the initial conditions or in the Hamiltonian. This question has been generally overlooked.

g) Classical mechanics assumes that it is reasonable to speak of two trajectories that may be infinitely close together. If then there is a finite difference on a certain quantity between these trajectories, it means a real discontinuity and a non-analytic function.

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[9] Brillouin, L.: Vie, Matière et Observation, Chap. V. Paris: Albin Michel 1959. — Information and Control, vol. 2, p. 45.

Wave-mechanics, applied to a stable problem, gives proper ψ functions, defined by quantum numbers. These functions (or the corresponding semi-classical trajectories) can never be infinitely close; they remain at finite intervals, and all quantities normally exhibit finite differences.

The transition from wave mechanics to classical mechanics is obtained by the *B. W. K. method*, but it should be emphasized that this method has *been discussed only for problems with separated variables.* The transition from wave mechanics to classical mechanics for non-separated variables is an open question. There are definite indications[10] that in some problems (scattering, collisions) the transition is not a smooth one and that geometrical optics, for instance, does not result from wave optics even for infinitely small wave lengths. There is a whole field of unsolved problems in this connection.

The research presented here was done under Contract Nonr 266(56) and was first described in a Quarterly Report dated July 31, 1959.

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[10] Brillouin, L.: J. of Applied Physics **20**, 1110 (1949).

Columbia University
New York City

(Received December 27, 1959)

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Druck der Universitätsdruckerei H. Stürtz AG., Würzburg
Printed in Germany